THE ELECTRIC MELVIN SOLUTION IN STRING THEORY

By

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Abstract

An electric version of the well-known magnetic Melvin solution of closed string theory is derived. By analogy with the Kaluza-Klein Melvin solution, which is flat space with points identified under a simultaneous rotation and translation in a compact dimension, an orbifold of Minkowski space involving identifications under a Lorentz boost and a translation is introduced. When dimensional reduction to 9 dimensions is performed, the resulting background involves an electric Kaluza-Klein gauge field, giving rise to the electric Melvin interpretation. As was done by other authors for the magnetic Melvin background, a curved generalization of this orbifold is derived using a series of T-duality transformations. The closed string is quantized on the resulting space, and the string spectrum and partition function are calculated.

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Chapter 1
Introduction

Before string theory can become a complete description of nature on the most fundamental level, the behaviour of strings in background electromagnetic fields must be understood. One way to introduce electromagnetism into string theory is to add boundary terms to the open string worldsheet action which couple charges at the string endpoints to a background gauge field. This is only an approximation valid for weak fields, however, since it does not take into account the effect of the energy of the gauge field on the curvature of the background spacetime. Moreover, since it does not change the bulk string worldsheet action, it does not include the effect of such a field on closed strings. It is difficult, however, to include these effects, since this involves finding a conformally invariant string theory background by solving the equations of string theory to all orders in perturbation theory, and few solutions are known.

One such solution is the Melvin background. Classically, there exists a class of static, cylindrically symmetric solutions of the Einstein-Maxwell system of equations involving electric and magnetic fields, known as Melvin solutions. An example of such a solution has a line element and electromagnetic field strength given by [6]:

$$\begin{aligned} ds^2 &= \left(1+\frac{B^2r^2}{4}\right)^2\left(-dt^2+dz^2+dr^2\right)+\frac{r^2}{\left(1+\frac{B^2r^2}{4}\right)^2}d\varphi^2 \qquad (1.1)\\ F &= \frac{Brdr\wedge d\varphi}{\left(1+\frac{B^2r^2}{4}\right)^2} \end{aligned}$$

where r and φ are polar coordinates in the x-y plane. This represents a magnetic flux tube along the z-axis, whose energy induces a cylindrically-symmetric spacetime curvature which confines the flux so that it is most intense at the z-axis, and goes to zero at infinity. A string theory analogue of this background can be obtained by considering the following metric:

$$ds_{10}^2 = -dt^2 + dx_s^2 + dx_9^2 + dr^2 + r^2\left(d\varphi + bd\dot{x}_9\right)^2 \qquad (1.2)$$

where x_s represents a number of flat spatial dimensions, x_9 is a compact dimension, and b is a constant. This is related to Minkowski space by the shift of the angular coordinate $\varphi \to \varphi + bdx_9$. By completing the square, the metric, (1.2), may be rewritten in the form:

$$\begin{aligned} ds_{10}^2 &= -dt^2 + dx_s^2 + dr^2 + \left(1 + b^2 r^2\right) \left(dx_9 + \frac{br^2}{1 + b^2 r^2} d\varphi\right)^2 \\ &+ \frac{r^2}{1 + b^2 r^2} d\varphi^2 \end{aligned} \tag{1.3}$$

In this form, dimensional reduction may be carried out along the x_9 direction (see appendix A), giving the 9-dimensional background:

$$\begin{aligned} ds_9^2 &= -dt^2 + dx_s^2 + dr^2 + \frac{r^2}{1 + b^2 r^2} d\varphi^2 \\ A_\varphi &= \frac{br^2}{1 + b^2 r^2} \qquad\qquad e^{2\sigma} = 1 + b^2 r^2 \end{aligned} \tag{1.4}$$

where A_φ is the Kaluza-Klein one-form gauge field, and $e^{2\sigma}$ is the Brans-Dicke scalar. The field strength, dA, generated by A_φ is that of a magnetic flux tube perpendicular to the $r - \varphi$ plane, with the magnitude of the field strength determined by the constant b. Thus, (1.4) is a Melvin-type cylindrically-symmetric spacetime background involving a magnetic field which is generated by a Kaluza-Klein gauge field. The ten-dimensional string theory metric, (1.2), from which it was obtained is called the Kaluza-Klein Melvin (KK Melvin) background.

The KK Melvin model has improved our understanding of strings in magnetic fields, and has also revealed new connections between different string theories. Because it is just flat space with a coordinate shift, the Melvin background (1.2) is an exact solution of string theory. Moreover, all string theories, including closed string theories, can be quantized on this space, and their mass spectra and partition functions calculated. Closed strings couple to the Kaluza-Klein gauge field via their Kaluza-Klein momentum and winding excitations, resulting in a non-trivial modification of their flat space spectrum. Also, it can be shown that, for certain values of the background parameters, type II string theory on Melvin space interpolates between type II string theory and the non-supersymmetric type 0 string theories on ordinary Minkowski space as the radius of the compact dimension is varied from 0 to ∞ [4]. This has led to the speculation that the type II superstring theories are the endpoint of tachyon condensation in the type 0 theories [7]. Similar connections have also been found between non-supersymmetric and supersymmetric heterotic string theories on Melvin space [8].

Another interesting feature of the background (1.2) is that it can be generalized to a curved background, on which string theory is also exactly solvable [2]. Performing a T-duality transformation along the x_9 direction (see Appendix B), (1.2) becomes

$$\begin{aligned} ds_{10}^2 &= -dt^2 + dx_s^2 + dx_9^2 + \frac{r^2}{1+b^2r^2}\left[d\varphi + bdx_9\right]\left[d\varphi - bdx_9\right] \\ B_{\varphi x_9} &= \frac{br^2}{1+b^2r^2} \qquad e^{2(\Phi-\Phi_0)} = \frac{1}{1+b^2r^2} \end{aligned} \tag{1.5}$$

where $B_{\varphi x_9}$ is an NS-NS antisymmetric tensor field and Φ is the dilaton. A second parameter $\tilde{b}$ can be introduced with the coordinate shift $\varphi \to \varphi + \tilde{b}x_9$. This parameter is analogous to the field strength parameter b in the T-dualized space. Performing this shift, the background (1.5) becomes

$$\begin{aligned} ds_{10}^2 &= -dt^2 + dx_s^2 + dx_9^2 + \frac{r^2}{1+b^2r^2}\left[d\varphi + \left(b+\tilde{b}\right)dx_9\right]\left[d\varphi + \right. \\ &\quad \left.\left(\tilde{b}-b\right)dx_9\right] \\ B_{\varphi x_9} &= \frac{br^2}{1+b^2r^2} \qquad e^{2(\Phi-\Phi_0)} = \frac{1}{1+b^2r^2} \end{aligned} \tag{1.6}$$

Another T-duality transformation along x_9 interchanges b and its T-dual $\tilde{b}$, giving

$$\begin{aligned} ds_{10}^2 &= -dt^2 + dx_s^2 + dx_9^2 + \frac{r^2}{1+\tilde{b}^2r^2}\left[d\varphi + \left(\tilde{b}+b\right)dx_9\right]\left[d\varphi + \right. \\ &\quad \left.\left(b-\tilde{b}\right)dx_9\right] \\ B_{\varphi x_9} &= \frac{\tilde{b}r^2}{1+\tilde{b}^2r^2} \qquad e^{2(\Phi-\Phi_0)} = \frac{1}{1+\tilde{b}^2r^2} \end{aligned} \tag{1.7}$$

This is a generalization of the KK Melvin background, since the choice $\tilde{b} = 0$ yields (1.2). Dimensional reduction of (1.7) to nine dimensions gives:

$$ds_9^2 = -dt^2 + dx_s^2 + dr^2 + \frac{r^2}{(1+b^2r^2)\left(1+\tilde{b}^2r^2\right)}d\varphi^2 \tag{1.8}$$

$$A_\varphi = \frac{br^2}{1+b^2r^2} \qquad B_\varphi = \frac{-\tilde{b}r^2}{1+\tilde{b}^2r^2} \qquad e^{2(\Phi-\Phi_0)} = \frac{1}{1+\tilde{b}^2r^2}$$

$$e^{2\sigma} = \frac{1+b^2r^2}{1+\tilde{b}^2r^2}$$

where A_φ is a Kaluza-Klein gauge field and B_φ is a gauge field arising from the dimensional reduction of the NS-NS 2-form. This 9-dimensional background is a generalization of the background (1.4) involving two magnetic gauge fields, with field strength parameters b and $\tilde{b}$. Thus the string theory background (1.7) is a Melvin background which generalizes the KK Melvin model. It can be shown that (1.7), like (1.2), is also a solution of string theory to all orders in α', and that it admits an exact solution of closed string theory [2][9].

In addition to magnetic backgrounds such as (1.1), there also exist similar solutions of the classical Einstein-Maxwell equations involving electric fields, and it would be useful to find electric Melvin solutions in string theory as well. The study of an electric Melvin background may improve our understanding of aspects of string theory in electric fields, such as Schwinger pair creation of strings. Moreover, it has been suggested that such backgrounds may have implications for string cosmology [11][12].

The purpose of this thesis is to construct an electric Melvin background analogous to (1.7), and to study closed string theory on this background. The background studied is a generalization of the one described in [11] to a curved space involving two electric field parameters. In chapter 2, the quantization of closed string theory on (1.7) is reviewed. This will illustrate the method that will be used to quantize the string on the electric Melvin background, and will also allow a comparison of the magnetic and electric cases. In chapter 3, the electric Melvin background is derived, and the geometry of the resulting space is described. The closed string is then quantized on this space, and the partition function is calculated.

Chapter 2
The Magnetic Melvin Solution

The pupose of this chapter is to review the quantization of the closed string on the magnetic Melvin background (1.7), as given in [2] and [3], in order to provide the necessary background for an understanding of the electric Melvin solution described in the following chapter. In section 2.1, the closed bosonic string is quantized on this background, and its quantum Hamiltonian is derived in terms of free string oscillator modes. The following section will extend these results to the type II superstring.

2.1 The Closed Bosonic String

2.1.1 Solution of the Equations of Motion

The bosonic string worldsheet Lagrangian for the background (1.7) can be obtained from the usual worldsheet action [5]:

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\left[\sqrt{h}\partial_\alpha X^\mu\partial_\beta X^\nu h^{\alpha\beta}g_{\mu\nu}+\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}\right. \\ &\quad \left.+\alpha'\sqrt{h}R\Phi\left(X\right)\right] \end{aligned} \tag{2.1}$$

Choosing the conformal gauge, the worldsheet metric $h_{\alpha\beta}$ becomes the flat Minkowski metric and the worldsheet Ricci scalar R is zero, giving

$$S = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\left[\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}+\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}\right] \tag{2.2}$$

Substituting the spacetime metric and antisymmetric tensor from (1.7) into (2.2) gives, for the Lagrangian,

$$\begin{aligned} L &= \partial_+r\partial_-r+\frac{r^2}{1+\tilde{b}^2r^2}\left[\partial_+\varphi+(b+\tilde{b})\partial_+x_9\right]\left[\partial_-\varphi+(b-\tilde{b})\partial_-x_9\right] \\ &\quad +\partial_+x_9\partial_-x_9-\partial_+t\partial_-t \end{aligned} \tag{2.3}$$

where the derivatives ∂_+ and ∂_- are with respect to the light cone worldsheet coordinates $\sigma_+ = \tau + \sigma$ and $\sigma_- = \tau - \sigma$. The flat space dimensions, x_s, were omitted.

In order to solve string theory on the Melvin background, a coordinate system must be chosen such that the equations of motion can be solved and the Hamiltonian is diagonal. Introducing light-cone coordinates

$$\begin{aligned} u &= x_9 - t \\ v &= x_9 + t \end{aligned} \tag{2.4}$$

and using coordinates $x = re^{i\varphi}$ and $x^* = re^{-i\varphi}$ for the $r - \varphi$ plane, (2.3) can be rewritten in the form:

$$\begin{aligned} L &= \partial_+ r \partial_- r + \frac{xx^*}{1 + \tilde{b}^2 xx^*} \left[\partial_+\varphi + b\partial_+ x_9 + \tilde{b}\partial_+(u+t)\right] \\ & \left[\partial_-\varphi + b\partial_- x_9 - \tilde{b}\partial_-(v-t)\right] + \partial_+ u \partial_- v \end{aligned} \tag{2.5}$$

In (2.5), the term

$$\partial_+ x_9 \partial_- t - \partial_+ t \partial_- x_9 \tag{2.6}$$

was added to the Lagrangian. This term does not affect the equations of motion, since it can be converted to a surface term by integrating by parts. The coordinate transformation

$$\varphi' = \varphi + bx_9 + \tilde{b}t \tag{2.7}$$

removes the explicit dependence of the Lagrangian on the parameter b and expresses it entirely in terms of light-cone coordinates, giving

$$L = \partial_+ r \partial_- r + \frac{xx^*}{1 + \tilde{b}^2 xx^*} \left[\partial_+\varphi' + \tilde{b}\partial_+ u\right] \left[\partial_-\varphi' - \tilde{b}\partial_- v\right] + \partial_+ u \partial_- v \tag{2.8}$$

The term involving $\partial_+ r \partial_- r$ in (2.8) can be combined with the term involving $\partial_+\varphi' \partial_-\varphi'$ to form the phase shifted x-coordinate

$$x' = re^{i\varphi'} = e^{i(bx_9 + \tilde{b}t)} x \tag{2.9}$$

Expanding (2.8) and obtaining a common denominator,

$$L = \frac{1}{1 + \tilde{b}^2 x' x'^*} \left[\partial_+ r \partial_- r + \tilde{b}^2 x' x'^* \partial_+ r \partial_- r\right] + \frac{x' x'^*}{1 + \tilde{b}^2 x' x'^*} \left[\partial_+\varphi' \partial_-\varphi' - \right.$$

$$\tilde{b}\partial_+\varphi'\partial_-v + \tilde{b}\partial_+u\partial_-\varphi' - \tilde{b}^2\partial_+u\partial_-v\Big] + \frac{1}{1+\tilde{b}^2x'x'^*}\left[\partial_+u\partial_-v + \right.$$
$$\left.\tilde{b}^2x'x'^*\partial_+u\partial_-v\right]$$
$$= \frac{1}{1+\tilde{b}^2x'x'^*}\Big[\partial_+r\partial_-r + \tilde{b}^2x'x'^*\partial_+r\partial_-r + x'x'^*\partial_+\varphi'\partial_-\varphi' -$$
$$\tilde{b}x'x'^*\partial_+\varphi'\partial_-v + \tilde{b}x'x'^*\partial_+u\partial_-\varphi' + \partial_+u\partial_-v\Big] \tag{2.10}$$

The first and third terms in (2.10) can be combined using

$$\begin{aligned} &\partial_+x'\partial_-x'^* \\ &= (\partial_+r + ir\partial_+\varphi')(\partial_-r - ir\partial_-\varphi') \\ &= \partial_+r\partial_-r + r^2\partial_+\varphi'\partial_-\varphi' + ir(\partial_+\varphi'\partial_-r - \partial_-\varphi'\partial_+r) \end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10) gives

$$\begin{aligned} L = &\frac{1}{1+\tilde{b}^2x'x'^*}\left[\partial_+x'\partial_-x'^* - ir(\partial_+\varphi'\partial_-r - \partial_-\varphi'\partial_+r) + \right. \\ &\tilde{b}^2x'x'^*\partial_+r\partial_-r - \tilde{b}x'x'^*\partial_+\varphi'\partial_-v + \tilde{b}x'x'^*\partial_+u\partial_-\varphi' + \\ &\left.\partial_+u\partial_-v\right] \end{aligned} \tag{2.12}$$

The second term in (2.12) is a total derivative and can be neglected:

$$\begin{aligned} &\frac{ir}{1+\tilde{b}^2r^2}(\partial_+\varphi'\partial_-r - \partial_-\varphi'\partial_+r) \\ = &\frac{i}{2\tilde{b}^2}\left\{\partial_-\left[\partial_+\varphi' ln\left(1+\tilde{b}^2r^2\right)\right] - \partial_+\left[\partial_-\varphi' ln\left(1+\tilde{b}^2r^2\right)\right]\right\} \end{aligned}$$

Therefore, from (2.11) and (2.12),

$$\begin{aligned} L = &\frac{1}{1+\tilde{b}^2x'x'^*}\left[\partial_+x'\partial_-x' - \tilde{b}^2x'x'^*\partial_+x'\partial_-x'^* + \tilde{b}^2x'x'^*\partial_+x'\partial_-x'^* + \right. \\ &\left.\tilde{b}^2x'x'^*\partial_+r\partial_-r - \tilde{b}x'x'^*\partial_+\varphi'\partial_-v + \tilde{b}x'x'^*\partial_-\varphi'\partial_+u + \partial_+u\partial_-v\right] \\ = &\partial_+x'\partial_-x'^* + \frac{1}{1+\tilde{b}^2x'x'^*}\left[-\tilde{b}^2(x'x'^*)^2\partial_+\varphi'\partial_-\varphi' - \tilde{b}x'x'^*\partial_+\varphi'\partial_-v\right. \\ &\left.+\tilde{b}x'x'^*\partial_-\varphi'\partial_+u + \partial_+u\partial_-v\right] \\ = &\partial_+x'\partial_-x'^* + \frac{1}{1+\tilde{b}^2x'x'^*}\left[\partial_+u - \tilde{b}x'x'^*\partial_+\varphi'\right]\left[\partial_-v + \right. \\ &\left.\tilde{b}x'x'^*\partial_-\varphi'\right] \end{aligned} \tag{2.13}$$

Using (2.9), φ' can be expressed in terms of x':

$$\varphi' = \frac{1}{2i} ln\left(\frac{x'}{x'^*}\right) \tag{2.14}$$

This gives

$$\partial_\pm \varphi' = \frac{1}{2i}\left[\frac{1}{x'}\partial_\pm x' - \frac{1}{x'^*}\partial_\pm x'^*\right] \tag{2.15}$$

Substituting (2.15) into (2.13),

$$\begin{aligned} L &= \frac{1}{1+\tilde{b}^2 x' x'^*}\left[\partial_+ u - \frac{\tilde{b}}{2i}\left(x'^*\partial_+ x' - x'\partial_+ x'^*\right)\right]\cdot \\ &\left[\partial_- v + \frac{\tilde{b}}{2i}\left(x'^*\partial_- x' - x'\partial_- x'^*\right)\right] + \partial_+ x' \partial_- x'^* \end{aligned} \tag{2.16}$$

The equations of motion of the Lagrangian in the form (2.16) can be solved in terms of free fields. Taking $F(x') = \frac{1}{1+\tilde{b}^2 x' x'^*}$ and $A_\pm = \frac{1}{2i}\left(x'^*\partial_\pm x' - x'\partial_\pm x'^*\right)$, the equations of motion for u and v are

$$\partial_-[F(x')(\partial_+ u - \tilde{b}A_+)] = 0 \tag{2.17}$$

$$\partial_+[F(x')(\partial_- v + \tilde{b}A_-)] = 0 \tag{2.18}$$

Integrating (2.17) and (2.18) once with respect to σ_- and σ_+, respectively, gives

$$F(x')\left[\partial_+ u - \tilde{b}A_+\right] = h_+(\sigma_+) \tag{2.19}$$

$$F(x')\left[\partial_- v + \tilde{b}A_-\right] = h_-(\sigma_-) \tag{2.20}$$

where h_+ and h_- are arbitrary functions of σ_+ and σ_-, respectively. From (2.16), the equation of motion for x' is:

$$\partial_+ \frac{\partial L}{\partial(\partial_+ x')} + \partial_- \frac{\partial L}{\partial(\partial_- x')} - \frac{\partial L}{\partial x'} = 0$$

$$\partial_+\left[-\frac{\tilde{b}}{2i}x'^* h_- + \partial_- x'^*\right] + \partial_-\left[\frac{\tilde{b}}{2i}x'^* h_+\right] - \left[-\tilde{b}^2 x'^* h_+ h_- + \frac{\tilde{b}}{2i}\partial_+ x'^* h_- - \frac{\tilde{b}}{2i}\partial_- x'^* h_+\right] = 0$$

$$\partial_+\partial_- x'^* + i\tilde{b}h_-\partial_+ x'^* - i\tilde{b}h_+\partial_- x'^* + \tilde{b}^2 h_+ h_- x'^* = 0$$

$$\partial_+\partial_- x' - i\tilde{b}h_-\partial_+ x' + i\tilde{b}h_+\partial_- x' + \tilde{b}^2 h_+ h_- x' = 0 \tag{2.21}$$

It can be checked by substitution that the solution to (2.21) is

$$x' = e^{i\tilde{b}V_- - i\tilde{b}U_+} X \qquad (2.22)$$

where U_+ and V_- are arbitrary functions of σ_+ and σ_-, respectively, such that $h_+ = \partial_+ U_+$ and $h_- = \partial_- V_-$, and X satisfies the free string equation of motion $\partial_+\partial_- X = 0$. X can be expressed as

$$\begin{aligned} X &= X_+ + X_- \\ X_+ &= e^{2i\gamma\sigma_+}\chi_+ \\ X_- &= e^{-2i\gamma\sigma_-}\chi_- \end{aligned} \qquad (2.23)$$

where χ_+ and χ_- are single-valued free fields with oscillator expansions

$$\chi_+ = i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \tilde{a}_n exp(-2in\sigma_+) \qquad (2.24)$$

$$\chi_- = i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} a_n exp(-2in\sigma_-) \qquad (2.25)$$

and γ is chosen so that the physical coordinate, $x = re^{i\varphi}$, is single-valued:

$$x(\sigma + \pi, \tau) = x(\sigma, \tau) \qquad (2.26)$$

The light-cone coordinates can also be solved for in terms of the free field X. Substituting (2.22) into (2.19) and simplifying gives

$$\begin{aligned} &F(x')\left[\partial_+ u - \frac{\tilde{b}}{2i}\left(-i\tilde{b}\partial_+ UXX^* + X^*\partial_+ X - i\tilde{b}\partial_+ UXX^* - X\partial_+ X^*\right)\right] \\ &= \partial_+ U \\ &\partial_+ u = \partial_+ U(1 + \tilde{b}^2 XX^*) - \frac{1}{2}\tilde{b}^2\partial_+ UXX^* - \frac{1}{2}\tilde{b}^2\partial_+ UXX^* + \\ &\frac{1}{2}i\tilde{b}(X^*\partial_+ X - X\partial_+ X^*) \\ &\partial_+ u = \partial_+ U + \frac{1}{2}i\tilde{b}(X\partial_+ X^* - X^*\partial_+ X) \end{aligned} \qquad (2.27)$$

Similarly,

$$\partial_- v = \partial_- V_- - \frac{1}{2}i\tilde{b}\left(X\partial_- X^* - X^*\partial_- X\right) \qquad (2.28)$$

These equations can be integrated to give

$$u = U_+ + U_- - \tilde{b}\tilde{\varphi}$$
$$v = V_+ + V_- - \tilde{b}\tilde{\varphi} \tag{2.29}$$

where U_- and V_+ are arbitrary functions of σ_- and σ_+, respectively, and

$$\tilde{\varphi} = 2\pi\alpha' \left[J_-(\sigma_-) - J_+(\sigma_+)\right] + \frac{i}{2}\left(X_+X_-^* - X_+^*X_-\right) \tag{2.30}$$

J_+ and J_- are angular momentum currents in the $r - \varphi$ plane, given by:

$$J_\pm(\sigma_\pm) = \frac{i}{4\pi\alpha'}\int_0^{\sigma_\pm} d\sigma_\pm \left(X_\pm\partial_\pm X_\pm^* - X_\pm^*\partial_\pm X_\pm\right) \tag{2.31}$$

2.1.2 Boundary Conditions

The solutions (2.22) and (2.29) for the non-trivial spacetime coordinates must be supplemented by the periodic closed string boundary condition, which further constrains the form of these solutions. From (2.4) and (2.30), u, v, and $\tilde{\varphi}$ satisfy the boundary conditions

$$u(\sigma+\pi,\tau) = u(\sigma,\tau) + 2\pi wR$$
$$v(\sigma+\pi,\tau) = v(\sigma,\tau) + 2\pi wR \tag{2.32}$$
$$\tilde{\varphi}(\sigma+\pi,\tau) = \tilde{\varphi}(\sigma,\tau) - 2\pi\alpha' J$$

where w is the winding number in the compact x_9 direction, and $J = J_+(\pi) + J_-(\pi) = J_L + J_R$ is the angular momentum in the $r-\varphi$ plane. The conditions on u and v are a consequence of the identification $x_9 \equiv x_9 + 2\pi wR$ for a closed string on a compact space with winding number w. Information on the form of $U_\pm$ and $V_\pm$ can be obtained from (2.32). $U_\pm$ and $V_\pm$ can be expressed in terms of the usual mode expansions:

$$U_\pm = \sigma_\pm p_\pm^u + U_\pm' \tag{2.33}$$
$$V_\pm = \sigma_\pm p_\pm^v + V_\pm'$$

where $p_\pm^u$ and $p_\pm^v$ are constants, and $U_\pm'$ and $V_\pm'$ are periodic functions of $\sigma_\pm$. (2.32) gives the boundary condition for U_+ and U_-:

$$u\left(\sigma+\pi,\tau\right) - u\left(\sigma,\tau\right) = 2\pi wR$$
$$U_+\left(\sigma_+ + \pi\right) + U_-\left(\sigma_- - \pi\right) - U_+\left(\sigma_+\right) - U_-\left(\sigma_-\right) - \tilde{b}\left[\tilde{\varphi}\left(\sigma+\pi,\tau\right) - \tilde{\varphi}\left(\sigma,\tau\right)\right] = 2\pi wR$$
$$U_+\left(\sigma_+ + \pi\right) + U_-\left(\sigma_- - \pi\right) = U_+\left(\sigma_+\right) + U_-\left(\sigma_-\right) - \tilde{b}2\pi\alpha' J + 2\pi wR \tag{2.34}$$

Similarly, for $V_\pm$,

$$V_+(\sigma_+ + \pi) + V_-(\sigma_- - \pi) = V_+(\sigma_+) + V_-(\sigma_-) - \tilde{b}2\pi\alpha' J + 2\pi wR \tag{2.35}$$

From (2.34) and (2.35), $p^u_\pm$ and $p^v_\pm$ have a similar form to that of free string theory, except with a shift in the winding number term:

$$\begin{aligned} p^u_\pm &= \pm(wR - \alpha'\tilde{b}J) + p^u \\ p^v_\pm &= \pm(wR - \alpha'\tilde{b}J) + p^v \\ p^u &= \frac{1}{2}(s-p) \\ p^v &= \frac{1}{2}(s+p) \end{aligned} \tag{2.36}$$

where s and p are constants which will be shown to be related to the canonical energy and momentum of the string.

An expression for the phase, γ, in (2.23) can be found by imposing periodicity on the physical coordinate, x. The expression for x in terms of free fields as obtained from (2.9), (2.22), and (2.23) is

$$\begin{aligned} x &= re^{i\varphi} \\ &= \exp i\left(-bx_9 - \tilde{b}t + \tilde{b}V_- - \tilde{b}U_+\right)[\exp(2i\gamma\sigma_+)\chi_+ + \\ &\quad \exp(-2i\gamma\sigma_-)\chi_-] \end{aligned} \tag{2.37}$$

Therefore, using (2.33) and (2.36),

$$\begin{aligned} &x(\sigma+\pi,\tau) = \exp i\Big\{-b\left[x_9(\sigma,\tau) + 2\pi wR\right] - \tilde{b}t + \tilde{b}\left[V_-(\sigma_-) + \right. \\ &\pi\left(wR - \alpha'\tilde{b}J\right) - \frac{1}{2}\pi(s+p)\Big] - \tilde{b}\Big[U_+(\sigma_+) + \pi(wR - \alpha'\tilde{b}J) + \\ &\frac{1}{2}\pi(s-p)\Big]\Big\} \cdot [\exp(2i\gamma\sigma_+ + 2i\gamma\pi)\chi_+ + \exp(-2i\gamma\sigma_- + 2i\gamma\pi)\chi_-] \\ &= \exp i\left[-2b\pi wR - \tilde{b}\pi s + 2\pi\gamma\right]x(\sigma,\tau) \end{aligned} \tag{2.38}$$

To satisfy (2.26), γ must therefore be given by

$$\gamma = bwR + \frac{1}{2}\tilde{b}s \tag{2.39}$$

Because it occurs only in the phase factors in (2.23), γ is defined only modulo an integer, so the restriction

$$0 \leq \gamma \leq 1 \tag{2.40}$$

may be imposed.

2.1.3 Hamiltonian

The Hamiltonian can be found from the Fourier modes of the worldsheet energy-momentum tensor. The general expression for the energy-momentum tensor is [1]:

$$T_{\alpha\beta} = -4\pi\alpha' \frac{1}{\sqrt{h}} \frac{\delta S}{\delta h^{\alpha\beta}} \tag{2.41}$$

where S is the world sheet action given by (2.1). Substituting (2.1) into (2.41) gives:

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu} - \frac{1}{2} h_{\alpha\beta} h^{\alpha'\beta'} \partial_{\alpha'} x^\mu \partial_{\beta'} x^\nu g_{\mu\nu} \tag{2.42}$$

The contribution of the dilaton term vanished because of the identity $\frac{\delta}{\delta h^{\alpha\beta}} \int \sqrt{h} R d\sigma d\tau = 0$. The second term in (2.42) vanishes in the light-cone coordinate system, with coordinates σ_+ and σ_-, when $h^{\alpha\beta}$ is the Minkowski metric. Thus,

$$T_{++} = \partial_+ x^\mu \partial_+ x^\nu g_{\mu\nu} \tag{2.43}$$

$$T_{--} = \partial_- x^\mu \partial_- x^\nu g_{\mu\nu} \tag{2.44}$$

The Lagrangian, (2.16), has the form

$$L = h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu} \tag{2.45}$$

Comparing (2.45) with (2.43) and (2.44), the energy momentum tensor can be seen to be:

$$T_{\pm\pm} = \frac{1}{1+\tilde{b}^2 x' x'^*} \left[\partial_\pm u - \frac{\tilde{b}}{2i} \left(x'^* \partial_\pm x' - x' \partial_\pm x'^* \right) \right] \left[\partial_\pm v + \frac{\tilde{b}}{2i} \left(x'^* \partial_\pm x' - x' \partial_\pm x'^* \right) \right] + \partial_\pm x' \partial_\pm x'^* \tag{2.46}$$

Substituting the expressions (2.29) and (2.22) for u, v, and x', T_{++} can be put into a form which resembles the free string energy momentum tensor:

$$\begin{aligned} T_{++} &= \partial_+ U_+ \left[\partial_+ V_+ - \frac{i\tilde{b}}{2}(X^* \partial_+ X - X \partial_+ X^*) - \frac{i\tilde{b}}{2}\left(-i\tilde{b}\partial_+ U_+ X^* X \right.\right. \\ &\quad \left.\left. + X^* \partial_+ X - i\tilde{b}\partial_+ U_+ X X^* - X\partial_+ X^*\right)\right] + \left[-i\tilde{b}\partial_+ U_+ X + \partial_+ X\right] \cdot \\ &\quad \left[i\tilde{b}\partial_+ U_+ X^* + \partial_+ X^*\right] \\ &= \partial_+ U_+ \partial_+ V_+ + \partial_+ X \partial_+ X^* \end{aligned} \tag{2.47}$$

Similarly,

$$T_{--} = \partial_- U_- \partial_- V_- + \partial_- X \partial_- X^* \tag{2.48}$$

Expressing (2.47) and (2.48) in terms of free fields using (2.23) and (2.33),

$$\begin{aligned} T_{\pm\pm} &= \left(p^u_\pm + \partial_\pm U'\right)\left(p^v_\pm + \partial_\pm V'\right) + (\pm 2i\gamma\chi_\pm + \partial_\pm \chi_\pm) \cdot \\ &\quad \left(\mp 2i\gamma\chi^*_\pm + \partial_\pm \chi^*_\pm\right) \\ &= p^u_\pm p^v_\pm + p^u_\pm \partial_\pm V' + p^v_\pm \partial_\pm U' + \partial_\pm U' \partial_\pm V' + 4\gamma^2 \chi_\pm \chi^*_\pm \\ &\quad \pm 2i\gamma\chi_\pm \partial_\pm \chi^*_\pm \mp 2i\gamma \chi^*_\pm \partial_\pm \chi_\pm + \partial_\pm \chi_\pm \partial_\pm \chi^*_\pm \end{aligned} \tag{2.49}$$

By choosing the light-cone gauge, the nonzero oscillator modes of the light-cone coordinate u can be set equal to zero, thus eliminating these modes from the Hamiltonian. These modes correspond to $U'_\pm$ in (2.33). Taking $U'_\pm = 0$, (2.49) becomes

$$\begin{aligned} T_{\pm\pm} &= p^u_\pm p^v_\pm + p^u_\pm \partial_\pm V'_\pm \pm 2i\gamma(\chi_\pm \partial_\pm \chi^*_\pm - \chi^* \partial_\pm \chi) + 4\gamma^2 \chi_\pm \chi^*_\pm \\ &\quad + \partial_\pm \chi_\pm \partial_\pm \chi^*_\pm \end{aligned} \tag{2.50}$$

The Virasoro generators L_0 and $\tilde{L}_0$ are Fourier coefficients of T_{--} and T_{++}:

$$\begin{aligned} L_0 &= \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma T_{--} \\ \tilde{L}_0 &= \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma T_{++} \end{aligned} \tag{2.51}$$

Substituting (2.50) into these expressions and using the oscillator expansions (2.24) and (2.25) gives:

$$L_0 = \frac{1}{4\pi\alpha'}\int_0^\pi \left[p^u_- p^v_- + p^u_- \partial_- V'_- - 2i\gamma(\chi_- \partial_- \chi^*_- - \chi^* \partial_- \chi)\right.$$

$$
\begin{aligned}
&+ \quad 4\gamma^2\chi_-\chi_-^* + \partial_-\chi_-\partial_-\chi_-^*\Big] \\
&= \frac{p_-^u p_-^v}{4\alpha'} - \frac{i\gamma}{4}\sum_{n=-\infty}^{\infty}\left[(2in)\, a_n^* a_n - (-2in)\, a_n^* a_n\right] + \frac{\gamma^2}{2}\sum_{n=-\infty}^{\infty} a_n^* a_n \\
&+ \frac{1}{8}\sum_{n=-\infty}^{\infty}(2in)(-2in)a_n^* a_n \\
&= \frac{p_-^u p_-^v}{4\alpha'} + \gamma\sum_{n=-\infty}^{\infty} n a_n^* a_n + \frac{1}{2}\gamma^2\sum_{n=-\infty}^{\infty} a_n^* a_n + \frac{1}{2}\sum_{n=-\infty}^{\infty} n^2 a_n^* a_n \\
&= \frac{p_-^u p_-^v}{4\alpha'} + \frac{1}{2}\sum_{n=-\infty}^{\infty}(n+\gamma)^2 a_n^* a_n \qquad (2.52)
\end{aligned}
$$

In the derivation of (2.52), the contribution of the V' term vanished by periodicity:

$$\int_0^\pi \partial_- V_-' d\sigma = -\int_0^\pi (\partial_+ - \partial_-)V_-' d\sigma = -\int_0^\pi \partial_\sigma V_-' d\sigma = -V_-'|_0^\pi = 0 \quad (2.53)$$

Similarly, it can be shown that

$$\tilde{L}_0 = \frac{p_+^u p_+^v}{4\alpha'} + \frac{1}{2}\sum_n (n-\gamma)^2\, \tilde{a}_n^* \tilde{a}_n \qquad (2.54)$$

The Hamiltonian is given by

$$H = L_0 + \tilde{L}_0 \qquad (2.55)$$

From (2.52), (2.54), and (2.36), this gives

$$
\begin{aligned}
H \;&=\; \frac{1}{4\alpha'}\Big[-(wR - \alpha'\tilde{b}J) + p^u\Big]\Big[-(wR - \alpha'\tilde{b}J) + p^v\Big] + \frac{1}{4\alpha'}\Big[(wR \\
&\quad -\alpha'\tilde{b}J\Big) + p^u\Big]\Big[(wR - \alpha'\tilde{b}J) + p^v\Big] + \frac{1}{2}\sum_{n=-\infty}^{\infty}(n+\gamma)^2 a_n^* a_n + \\
&\quad \frac{1}{2}\sum_{n=-\infty}^{\infty}(n-\gamma)^2\, \tilde{a}_n^* \tilde{a}_n \\
&= \frac{1}{2\alpha'}(wR - \alpha'\tilde{b}J)^2 + \frac{1}{2\alpha'}p^u p^v + \frac{1}{2}\sum_{n=-\infty}^{\infty}(n+\gamma)^2 a_n^* a_n \\
&\quad + \frac{1}{2}\sum_{n=-\infty}^{\infty}(n-\gamma)^2\, \tilde{a}_n^* \tilde{a}_n
\end{aligned}
$$

$$\begin{aligned} &= \frac{w^2R^2}{2\alpha'} - wR\tilde{b}J + \frac{\alpha'}{2}\tilde{b}^2J^2 + \frac{1}{8\alpha'}\left(s^2-p^2\right) + \frac{1}{2}\sum_{n=-\infty}^{\infty}(n+\gamma)^2 a_n^* a_n \\ &\quad + \frac{1}{2}\sum_{n=-\infty}^{\infty}(n-\gamma)^2\tilde{a}_n^*\tilde{a}_n \\ &= \frac{1}{8\alpha'}\left(4w^2R^2+s^2-p^2\right) + \frac{1}{2}\sum_n (n+\gamma)^2 a_n^* a_n \\ &\quad + \frac{1}{2}\sum_n (n-\gamma)^2\tilde{a}_n^*\tilde{a}_n - \tilde{b}wRJ + \frac{1}{2}\alpha'\tilde{b}^2J^2 \end{aligned} \tag{2.56}$$

The contribution of the additional 22 flat space coordinates to (2.56) is the same as that of free bosonic string theory, and was omitted in the derivation of (2.56).

The parameters s and p are related to the energy of the string and its momentum in the x_9 direction. The canonical energy and momentum are given by:

$$E = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \frac{\partial L}{\partial(\partial_\tau t)} \tag{2.57}$$

$$p_{x_9} = \frac{m}{R} = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \frac{\partial L}{\partial(\partial_\tau x_9)} \tag{2.58}$$

where m is the momentum number in the x_9 direction and R is the compactification radius. From (2.3), the integrand of (2.57) is just given by the free theory expression:

$$\frac{\partial L}{\partial(\partial_\tau t)} = -\partial_\tau t \tag{2.59}$$

The integral can be carried out using (2.4), (2.29), and (2.33):

$$\begin{aligned} E &= -\frac{1}{2\pi\alpha'}\int_0^\pi \partial_\tau t d\sigma \\ &= -\frac{1}{2\pi\alpha'}\int_0^\pi \partial_\tau\left[\frac{1}{2}(v-u)\right]d\sigma \\ &= -\frac{1}{4\pi\alpha'}\int_0^\pi \partial_\tau\left[V_+ + V_- - U_+ - U_-\right] \\ &= -\frac{1}{4\pi\alpha'}\int_0^\pi \partial_\tau\Big[(\tau+\sigma)(wR-\tilde{b}\alpha'J) + \frac{1}{2}(\tau+\sigma)(s+p) + V_+' \\ &\quad -(\tau-\sigma)(wR-\tilde{b}\alpha'J) + \frac{1}{2}(\tau-\sigma)(s+p) + V_-' - (\tau+\sigma)\cdot \end{aligned}$$

$$
\begin{aligned}
& (wR - \tilde{b}\alpha' J) - \frac{1}{2}(\tau+\sigma)(s-p) + (\tau-\sigma)(wR - \tilde{b}\alpha' J) \\
& -\frac{1}{2}(\tau-\sigma)(s-p)\Big] d\sigma \\
= \; & -\frac{1}{4\pi\alpha'} \int_0^\pi \partial_\tau \left[2p\tau + V'_+ + V'_-\right] d\sigma \\
= \; & -\frac{p}{2\alpha'}
\end{aligned} \tag{2.60}
$$

The contribution of the last two terms vanished by periodicity of $V'_\pm$ in σ since, for example,

$$
\begin{aligned}
\partial_- V'_+ &= 0 \\
\partial_\tau V'_+ &= \partial_\sigma V'_+ \\
\int_0^\pi \partial_\tau V'_+ d\sigma &= \int_0^\pi \partial_\sigma V'_+ d\sigma = V'_+|_0^\pi = 0
\end{aligned}
$$

The expression for (2.58) is more complicated because of the nontrivial dependence of (2.3) on x_9. Using (2.3) to calculate the integrand gives:

$$
\begin{aligned}
\frac{\partial L}{\partial(\partial_\tau x_9)} &= \frac{r^2}{1+\tilde{b}^2 r^2}\frac{1}{2}(b+\tilde{b})\left[\partial_-\varphi + (b-\tilde{b})\partial_- x_9\right] \\
&+ \frac{r^2}{1+\tilde{b}^2 r^2}\frac{1}{2}(b-\tilde{b})\left[\partial_+\varphi + (b+\tilde{b})\partial_+ x_9\right] + \partial_\tau x_9
\end{aligned} \tag{2.61}
$$

The first two terms in (2.61) can be integrated using a relation between φ and $\tilde{\varphi}$, where $\tilde{\varphi}$ is the expression in (2.30). Using (2.30) and (2.31),

$$
\begin{aligned}
\partial_+\tilde{\varphi} &= \partial_+\left[2\pi\alpha'\left[J_-(\sigma_-) - J_+(\sigma_+)\right] + \frac{i}{2}\left(X_+X_-^* - X_+^*X_-\right)\right] \\
&= -\frac{i}{2}\left(X_+\partial_+X_+^* - X_+^*\partial_+X_+\right) - \frac{i}{2}\left(X_-\partial_+X_+^* - X_-^*\partial_+X_+\right) \\
&= \frac{i}{2}\left(X^*\partial_+X - X\partial_+X_+^*\right)
\end{aligned} \tag{2.62}
$$

This can be expressed in terms of the coordinates r, φ, and x_9 using (2.9) and (2.22):

$$
\begin{aligned}
\partial_+\tilde{\varphi} &= \frac{i}{2}r\left[\partial_+ r + ir\left(\partial_+\varphi + b\partial_+x_9 + \tilde{b}\partial_+ t + \tilde{b}\partial_+ U_+\right)\right] - \\
&\quad \frac{i}{2}r\left[\partial_+ r - ir\left(\partial_+\varphi + b\partial_+x_9 + \tilde{b}\partial_+ t + \tilde{b}\partial_+ U_+\right)\right]
\end{aligned} \tag{2.63}
$$

From (2.19), the last term is given by

$$\begin{aligned}
\partial_+ U &= F(x')(\partial_+ u - \tilde{b} A_+) \\
&= \frac{1}{1+\tilde{b}^2 r^2}\left\{\partial_+ x_9 - \partial_+ t - \frac{\tilde{b}}{2i}\left[r\left(\partial_+ r + ir\left(\partial_+\varphi + b\partial_+ x_9 + \tilde{b}\partial_+ t\right)\right)\right.\right. \\
&\quad \left.\left. -r\left(\partial_+ r - ir\left(\partial_+\varphi + b\partial_+ x_9 + \tilde{b}\partial_+ t\right)\right)\right]\right\} \\
&= \frac{1}{1+\tilde{b}^2 r^2}\left[\partial_+ x_9 - \partial_+ t - \tilde{b} r^2\left(\partial_+\varphi + b\partial_+ x_9 + \tilde{b}\partial_+ t\right)\right] \qquad (2.64)
\end{aligned}$$

Substituting (2.64) into (2.63),

$$\begin{aligned}
\partial_+\tilde{\varphi} &= -r^2\left\{\partial_+\varphi + b\partial_+ x_9 + \tilde{b}\partial_+ t + \frac{\tilde{b}}{1+\tilde{b}^2 r^2}\left[\partial_+ x_9 - \right.\right. \\
&\quad \left.\left. \partial_+ t - \tilde{b} r^2\left(\partial_+\varphi + b\partial_+ x_9 + \tilde{b}\partial_+ t\right)\right]\right\} \\
&= -r^2\left\{\partial_+\varphi + b\partial_+ x_9 + \frac{\tilde{b}}{1+\tilde{b}^2 r^2}\left[\partial_+ x_9 - \tilde{b} r^2\left(\partial_+\varphi + b\partial_+ x_9\right)\right]\right\} \\
&= -r^2\left[\partial_+\varphi + b\partial_+ x_9 + \frac{\tilde{b}}{1+\tilde{b}^2 r^2}\partial_+ x_9 - \frac{\tilde{b}^2 r^2}{1+\tilde{b}^2 r^2}\partial_+\varphi\right. \\
&\quad \left. -\frac{\tilde{b}^2 b r^2}{1+\tilde{b}^2 r^2}\partial_+ x_9\right] \\
&= -\frac{r^2}{1+\tilde{b}^2 r^2}\left[\partial_+\varphi + \tilde{b}^2 r^2\partial_+\varphi + b\partial_+ x_9 + b\tilde{b}^2 r^2\partial_+ x_9 + \tilde{b}\partial_+ x_9 - \right. \\
&\quad \left. \tilde{b}^2 r^2\partial_+\varphi - b\tilde{b}^2 r^2\partial_+ x_9\right] \\
&= -\frac{r^2}{1+\tilde{b}^2 r^2}\left[\partial_+\varphi + (b+\tilde{b})\partial_+ x_9\right] \qquad (2.65)
\end{aligned}$$

Similarly, it can be shown that

$$\partial_-\tilde{\varphi} = \frac{r^2}{1+\tilde{b}^2 r^2}\left[\partial_-\varphi + (b-\tilde{b})\partial_- x_9\right] \qquad (2.66)$$

Using (2.65) and (2.66), (2.61) becomes

$$\begin{aligned}
\frac{\partial L}{\partial(\partial_\tau x_9)} &= \partial_\tau x_9 - (b-\tilde{b})\partial_+\tilde{\varphi} + (b+\tilde{b})\partial_-\tilde{\varphi} \\
&= \partial_\tau x_9 - \frac{1}{2}(b-\tilde{b})(\partial_\tau + \partial_\sigma)\tilde{\varphi} + \frac{1}{2}(b+\tilde{b})(\partial_\tau - \partial_\sigma)\tilde{\varphi} \\
&= \partial_\tau x_9 - b\partial_\sigma\tilde{\varphi} + \tilde{b}\partial_\tau\tilde{\varphi} \qquad (2.67)
\end{aligned}$$

The first term in (2.67) can be evaluated using (2.4), (2.29) and (2.33):

$$\begin{aligned}
\partial_\tau x_9 &= \partial_\tau \left[\frac{1}{2}(u+v)\right] \\
&= \frac{1}{2}\partial_\tau \Big[2(\tau+\sigma)(wR-\alpha'\tilde{b}J) + \frac{1}{2}(\tau+\sigma)(s-p) - 2(\tau-\sigma)\cdot \\
&\quad (wR-\alpha'\tilde{b}J) + \frac{1}{2}(\tau-\sigma)(s-p) + \frac{1}{2}(\tau+\sigma)(s+p) + \frac{1}{2}(\tau-\sigma)\cdot \\
&\quad (s+p) + V'_+ + V'_- - 2\tilde{b}\tilde{\varphi}\Big] \\
&= s - \tilde{b}\partial_\tau\tilde{\varphi} + \partial_\tau V'_+ + \partial_\tau V'_- \qquad (2.68)
\end{aligned}$$

As in (2.60), the last two terms integrate out. Substituting (2.67) and (2.68) gives, for the momentum (2.58),

$$p_{x_9} = \frac{1}{2\pi\alpha'}\int_0^\pi (s - b\partial_\sigma\tilde{\varphi})\, d\sigma \qquad (2.69)$$

Using (2.30), the second term is

$$\begin{aligned}
&\int_0^\pi \partial_\sigma\tilde{\varphi} d\sigma \\
&= 2\pi\alpha'\left[J_-(\sigma_-) - J_+(\sigma_+)\right]|_{\sigma=0}^{\sigma=\pi} + \frac{i}{2}\left(X_+X_-^* - X_+^*X_-\right)|_{\sigma=0}^{\sigma=\pi} \\
&= 2\pi\alpha'\left[J_-(\tau-\pi) - J_+(\tau+\pi) - J_-(\tau) + J_+(\tau)\right] \\
&= 2\pi\alpha'\left[J_-(-\pi) - J_+(\pi)\right] \\
&= -2\pi\alpha'\left[J_-(\pi) + J_+(\pi)\right] \\
&= -2\pi\alpha' J \qquad (2.70)
\end{aligned}$$

The momentum is therefore

$$p_{x_9} = \frac{m}{R} = \frac{1}{2\alpha'}(s + 2\alpha' bJ) \qquad (2.71)$$

Solving for s gives:

$$s = 2\alpha'\left[\frac{m}{R} - bJ\right] \qquad (2.72)$$

Substituting (2.72) into (2.39) yields the expression for γ:

$$\gamma = bRw + \frac{\tilde{b}\alpha' m}{R} - \alpha' b\tilde{b}J \qquad (2.73)$$

2.1.4 Light-Cone Gauge Quantization

Quantization of the string in the light-cone gauge can be carried out by introducing canonical commutation relations for the spacetime coordinates. For the cartesian coordinates x_1 and x_2 in the $r - \varphi$ plane, these are

$$[P^1(\sigma,\tau), x^1(\sigma',\tau)] = [P^2(\sigma,\tau), x^2(\sigma',\tau)] = -i\delta(\sigma-\sigma') \tag{2.74}$$

where P^1 and P^2 are the canonical momenta associated with x^1 and x^2. In terms of x and x^*, (2.74) becomes

$$[P_x(\sigma,\tau), x^*(\sigma',\tau)] = [P_x^*(\sigma,\tau), x(\sigma',\tau)] = -i\delta(\sigma-\sigma') \tag{2.75}$$

where $P_x = \frac{1}{2}(P_1 + iP_2)$ and $P_x^* = \frac{1}{2}(P_1 - iP_2)$.

The commutation relations of the Fourier modes can be derived from (2.75). The calculation can be simplified using a relation between the fields x and x^* and the free fields X and X^*. In [2] it is shown that the action associated with x and x^* is related to a free string action involving X and X^* via the duality that relates (1.2) to (1.7). This relation implies the equivalence of the canonical commutation relations (2.75) to those of the free fields,

$$[P_X(\sigma,\tau), X^*(\sigma',\tau)] = [P_X^*(\sigma,\tau), X(\sigma',\tau)] = -i\delta(\sigma-\sigma') \tag{2.76}$$

where, since the action of X is the free string one, $P_X = \frac{1}{4\pi\alpha'}\partial_\tau X$ is the usual canonical momentum from free string theory. Substituting the mode expansions (2.24) and (2.25) in the left-hand side of (2.76) gives

$$\begin{aligned}
&\frac{1}{4\pi\alpha'}\left[\partial_\tau X(\sigma,\tau), X^*(\sigma',\tau)\right] \\
=\ &\frac{1}{4\pi}\Bigg[e^{2i\gamma(\tau+\sigma)}\sum_{n=-\infty}^{\infty}(i\gamma - in)\,\tilde{a}_n e^{-2in(\tau+\sigma)} + e^{-2i\gamma(\tau-\sigma)}. \\
&\sum_{n=-\infty}^{\infty}(-i\gamma - in)\,a_n e^{-2in(\tau-\sigma)}, e^{-2i\gamma(\tau+\sigma)}\sum_{n=-\infty}^{\infty}\tilde{a}_n^* e^{2in(\tau+\sigma)} + \\
&e^{2i\gamma(\tau-\sigma)}\sum_{n=-\infty}^{\infty}a_n^* e^{2in(\tau-\sigma)}\Bigg] \\
=\ &\frac{-i}{4}e^{2i\gamma(\sigma-\sigma')}\Bigg[\frac{1}{\pi}\sum_{n=-\infty}^{\infty}e^{-2in(\sigma-\sigma')}\left[\tilde{a}_n, \tilde{a}_n^*\right](n-\gamma) + \\
&\frac{1}{\pi}\sum_{n=-\infty}^{\infty}e^{2in(\sigma-\sigma')}\left[a_n, a_n^*\right](n+\gamma)\Bigg]
\end{aligned} \tag{2.77}$$

Equating this to the right-hand side of (2.76), it can be seen that the Fourier modes must satisfy the commutation relations

$$[a_n, a_m^*] = 2\,(n+\gamma)^{-1}\,\delta_{nm} \tag{2.78}$$

$$[\tilde{a}_n, \tilde{a}_m^*] = 2\,(n-\gamma)^{-1}\,\delta_{nm} \tag{2.79}$$

These commutation relations show that the Hamiltonian depends on γ only modulo an integer, as expected from the fact that it is a phase. If an arbitrary integer, k, is added to γ, the commutation relations remain the same if the redefinitions $a_n \to a_{n+k}$, $a_n^* \to a_{n+k}^*$, $\tilde{a}_n \to \tilde{a}_{n-k}$, and $\tilde{a}_n^* \to \tilde{a}_{n-k}^*$ are made. The Hamiltonian is invariant under these redefinitions, since they leave the infinite sums over oscillators unchanged. The theory is therefore invariant under shifting γ by an integer, and γ may be restricted to the range $0 \le \gamma \le 1$. For γ in this range, the normalized creation and annihilation operators corresponding to the Fourier modes in (2.78) and (2.79) are given by:

$$\begin{aligned}
b_{n+}^\dagger &= \omega_- a_{-n} & b_{n+} &= \omega_- a_{-n}^* \\
b_{n-}^\dagger &= \omega_+ a_n^* & b_{n-} &= \omega_+ a_n \\
\tilde{b}_{n+}^\dagger &= \omega_+ \tilde{a}_{-n} & \tilde{b}_{n+} &= \omega_+ \tilde{a}_{-n}^* \\
\tilde{b}_{n-}^\dagger &= \omega_- \tilde{a}_n^* & \tilde{b}_{n-} &= \omega_- \tilde{a}_n \\
b_0^\dagger &= \sqrt{\frac{1}{2}\gamma} a_0^* & b_0 &= \sqrt{\frac{1}{2}\gamma} a_0 \\
\tilde{b}_0^\dagger &= \sqrt{\frac{1}{2}\gamma} \tilde{a}_0 & \tilde{b}_0 &= \sqrt{\frac{1}{2}\gamma} \tilde{a}_0^* \\
\omega_\pm &= \sqrt{\frac{1}{2}(n \pm \gamma)}, & n &= 1, 2, \ldots
\end{aligned} \tag{2.80}$$

where the b operators satisfy

$$\begin{aligned}
\left[b_{n\pm}, b_{m\pm}^\dagger\right] &= \delta_{nm}, & \left[\tilde{b}_{n\pm}, \tilde{b}_{m\pm}^\dagger\right] &= \delta_{nm} \\
\left[b_0, b_0^\dagger\right] &= 1, & \left[\tilde{b}_0, \tilde{b}_0^\dagger\right] &= 1
\end{aligned} \tag{2.81}$$

Using these quantized oscillator modes, an operator expression can be derived for the quantum Hamiltonian corresponding to the classical Hamiltonian (2.56). The quantized version of (2.52) and (2.54) is obtained by symmetrizing the sums over modes, giving

$$\frac{1}{2}\sum_n (n+\gamma)^2 a_n^* a_n = \frac{1}{4}\sum_n (n+\gamma)^2 (a_n^* a_n + a_n a_n^*) \tag{2.82}$$

$$\frac{1}{2}\sum_n (n-\gamma)^2 \tilde{a}_n^* \tilde{a}_n = \frac{1}{4}\sum_n (n-\gamma)^2 (\tilde{a}_n^* \tilde{a}_n + \tilde{a}_n \tilde{a}_n^*) \tag{2.83}$$

In terms of the normalized operators (2.81), (2.82) is

$$\begin{aligned} &\frac{1}{4}\sum_n (n+\gamma)^2 (a_n^* a_n + a_n a_n^*) \\ = &\frac{1}{2}\sum_{n=1}^{\infty} (n+\gamma)\left(b_{n-}^\dagger b_{n-} + b_{n-} b_{n-}^\dagger\right) + \frac{1}{2}\sum_{n=1}^{\infty} (n-\gamma) \cdot \\ &\left(b_{n+} b_{n+}^\dagger + b_{n+}^\dagger b_{n+}\right) + \frac{1}{2}\gamma\left(b_0^\dagger b_0 + b_0 b_0^\dagger\right) \end{aligned} \tag{2.84}$$

The infinite sums arising in the normal-ordering of this expression can be evaluated using a generalized zeta function regularization:

$$\sum_{n=1}^{\infty} (n+c) = -\frac{1}{12} + \frac{1}{2}c(1-c) \tag{2.85}$$

where c is an arbitrary constant. This gives, for (2.84),

$$\begin{aligned} &\frac{1}{4}\sum_n (n+\gamma)^2 (a_n^* a_n + a_n a_n^*) \\ = &\sum_{n=1}^{\infty} (n+\gamma) b_{n-}^\dagger b_{n-} - \frac{1}{24} + \frac{1}{4}\gamma(1-\gamma) + \sum_{n=1}^{\infty} (n-\gamma) \cdot \\ &b_{n+}^\dagger b_{n+} - \frac{1}{24} - \frac{1}{4}\gamma(1+\gamma) + \gamma b_0^\dagger b_0 + \frac{1}{2}\gamma \\ = &\sum_{n=1}^{\infty} (n+\gamma) b_{n-}^\dagger b_{n-} + \sum_{n=1}^{\infty} (n-\gamma) b_{n+}^\dagger b_{n+} + \gamma b_0^\dagger b_0 \\ &-\frac{1}{12} + \frac{1}{2}\gamma(1-\gamma) \end{aligned} \tag{2.86}$$

Similarly, for the left-movers,

$$\begin{aligned} &\frac{1}{4}\sum_n (n-\gamma)^2 (\tilde{a}_n^* \tilde{a}_n + \tilde{a}_n \tilde{a}_n^*) \\ = &\sum_{n=1}^{\infty} (n-\gamma) \tilde{b}_{n-}^\dagger \tilde{b}_{n-} + \sum_{n=1}^{\infty} (n+\gamma) \tilde{b}_{n+}^\dagger \tilde{b}_{n+} + \gamma \tilde{b}_0^\dagger \tilde{b}_0 \\ &-\frac{1}{12} + \frac{1}{2}\gamma(1-\gamma) \end{aligned} \tag{2.87}$$

Substituting the expressions (2.86) and (2.87) for the quantized version of the infinite sums in (2.56) gives the following expression for the quantum Hamiltonian:

$$\begin{aligned} H &= \frac{1}{8\alpha'}\left(4w^2R^2+s^2-p^2\right)+\sum_{n=1}^{\infty}(n+\gamma)\,b_{n-}^{\dagger}b_{n-}+ \\ & \sum_{n=1}^{\infty}(n-\gamma)\,b_{n+}^{\dagger}b_{n+}+\gamma b_0^{\dagger}b_0+\sum_{n=1}^{\infty}(n-\gamma)\,\tilde{b}_{n-}^{\dagger}\tilde{b}_{n-}+ \\ & \sum_{n=1}^{\infty}(n+\gamma)\,\tilde{b}_{n+}^{\dagger}\tilde{b}_{n+}+\gamma\tilde{b}_0^{\dagger}\tilde{b}_0-\tilde{b}wRJ-2+ \\ & \gamma(1-\gamma)+\frac{1}{2}\alpha'\tilde{b}^2J^2 \end{aligned} \tag{2.88}$$

The normal-ordering constant in (2.88) was obtained by adding the normal-ordering constants in (2.86) and (2.87) to the usual contributions from the remaining 22 trivial coordinates.

The infinite sums in (2.88) can be identified with level number and angular momentum operators. Right- and left-moving level number operators can be defined by

$$\begin{aligned} N_R &= \sum_{n=1}^{\infty} n\left(b_{n-}^{\dagger}b_{n-}+b_{n+}^{\dagger}b_{n+}\right) \\ N_L &= \sum_{n=1}^{\infty} n\left(\tilde{b}_{n-}^{\dagger}\tilde{b}_{n-}+\tilde{b}_{n+}^{\dagger}\tilde{b}_{n+}\right) \end{aligned} \tag{2.89}$$

Similar expansions can be obtained for the angular momentum operators J_L and J_R, using (2.31) and (2.23). For example, the classical expression for J_R is

$$\begin{aligned} J_R &= \frac{i}{4\pi\alpha'}\int_0^{\pi} d\sigma_-\left(X_-\partial_-X_-^*-X_-^*\partial_-X_-\right) \\ &= \frac{i}{4\pi\alpha'}\int_0^{\pi} d\sigma_-\left[\chi_-\left(2i\gamma\chi_-^*+\partial_-\chi_-^*\right)-\chi_-^*\left(-2i\gamma\chi_-+\partial_-\chi_-\right)\right] \\ &= \frac{i}{8}\left[\sum_{n=-\infty}^{\infty}(2i\gamma+2in)\,a_n^*a_n-\sum_{n=-\infty}^{\infty}(-2i\gamma-2in)\,a_n^*a_n\right] \\ &= -\frac{1}{2}\sum_{n=-\infty}^{\infty}(n+\gamma)a_n^*a_n \end{aligned} \tag{2.90}$$

Symmetrizing to obtain the quantum version,

$$\hat{J}_R = -\frac{1}{4}\sum_{n=-\infty}^{\infty}(n+\gamma)\left(a_n^* a_n + a_n a_n^*\right) \tag{2.91}$$

where a hat is used to distinguish this operator from its normal-ordered form. Similarly,

$$\hat{J}_L = -\frac{1}{4}\sum_{n=-\infty}^{\infty}(n-\gamma)\left(\tilde{a}_n^* \tilde{a}_n + \tilde{a}_n \tilde{a}_n^*\right) \tag{2.92}$$

Using (2.80) and normal ordering,

$$\begin{aligned}
\hat{J}_R &= -\frac{1}{2}\sum_{n=1}^{\infty}\left(b_{n-}^\dagger b_{n-} + b_{n-} b_{n-}^\dagger\right) + \frac{1}{2}\sum_{n=1}^{\infty}\left(b_{n+} b_{n+}^\dagger + b_{n+}^\dagger b_{n+}\right) \\
&\quad -\frac{1}{2}\left(b_0^\dagger b_0 + b_0 b_0^\dagger\right) \\
&= -\sum_{n=1}^{\infty} b_{n-}^\dagger b_{n-} - \zeta(0) + \sum_{n=1}^{\infty} b_{n+}^\dagger b_{n+} + \zeta(0) - b_0^\dagger b_0 - \frac{1}{2} \\
&= -\sum_{n=1}^{\infty} b_{n-}^\dagger b_{n-} + \frac{1}{2} + \sum_{n=1}^{\infty} b_{n+}^\dagger b_{n+} - \frac{1}{2} - b_0^\dagger b_0 - \frac{1}{2} \\
&= \sum_{n=1}^{\infty}\left(b_{n+}^\dagger b_{n+} - b_{n-}^\dagger b_{n-}\right) - b_0^\dagger b_0 - \frac{1}{2}
\end{aligned} \tag{2.93}$$

where zeta function regularization was used to compute the normal ordering constants. Similarly, it can be shown that

$$\hat{J}_L = \sum_{n=1}^{\infty}\left(\tilde{b}_{n+}^\dagger \tilde{b}_{n+} - \tilde{b}_{n-}^\dagger \tilde{b}_{n-}\right) + \tilde{b}_0^\dagger \tilde{b}_0 + \frac{1}{2} \tag{2.94}$$

The normal ordering constants of J_L and J_R cancel in the angular momentum operator $J = J_L + J_R$, giving a normal ordering constant of zero for J. The first and second terms in (2.93) are the spin, S_R, and the orbital angular momentum, or Landau level, l_R, respectively:

$$\hat{J}_R = -l_R + S_R - \frac{1}{2} \tag{2.95}$$

Similarly,

$$\hat{J}_L = l_L + S_L + \frac{1}{2} \tag{2.96}$$

Using (2.60), (2.72), (2.89), (2.93), and (2.94), the Hamiltonian, (2.88), can be expressed in terms of N and J:

$$\begin{aligned} H &= \frac{1}{8\alpha'}\left[4w^2R^2 + 4\alpha'^2\left(\frac{m}{R} - bJ\right)^2 - 4\alpha'^2E^2\right] + [N_R - \gamma J_R] \\ &\quad + [N_L + \gamma J_L] - \tilde{b}wRJ + \frac{1}{2}\alpha'\tilde{b}^2J^2 - 2 + \gamma(1-\gamma) \\ &= -\frac{1}{2}\alpha'E^2 + N_L + N_R + \frac{1}{2}\alpha'\left(\frac{m}{R} - bJ\right)^2 + \\ &\quad \frac{1}{2}\left(\frac{wR}{\sqrt{\alpha'}} - \sqrt{\alpha'}\tilde{b}J\right)^2 - \gamma(J_R - J_L) - 2 + \gamma(1-\gamma) \end{aligned} \tag{2.97}$$

This operator expression for the Hamiltonian must be supplemented by the level-matching condition, $L_0 - \tilde{L}_0 = 0$. From (2.52) and (2.54),

$$\begin{aligned} & L_0 - \tilde{L}_0 \\ &= \frac{p_-^u p_-^v}{4\alpha'} + \frac{1}{2}\sum_n (n+\gamma)^2 a_n^* a_n - \frac{p_+^u p_+^v}{4\alpha'} - \frac{1}{2}\sum_n (n-\gamma)^2 \tilde{a}_n^* \tilde{a}_n \\ &= \frac{1}{4\alpha'}\left[-\left(wR - \alpha'\tilde{b}J\right) + p^u\right]\left[-\left(wR - \alpha'\tilde{b}J\right) + p^v\right] - \frac{1}{4\alpha'} \cdot \\ &\quad \left[\left(wR - \alpha'\tilde{b}J\right) + p^u\right]\left[\left(wR - \alpha'\tilde{b}J\right) + p^v\right] + (N_R - \gamma J_R) \\ &\quad - (N_L + \gamma J_L) \\ &= -\frac{1}{2\alpha'}\left(wR - \alpha'\tilde{b}J\right)(p^u + p^v) + N_R - N_L - \gamma J \\ &= -\frac{1}{2\alpha'}\left(wR - \alpha'\tilde{b}J\right)s + N_R - N_L - \gamma J \\ &= -\left(wR - \alpha'\tilde{b}J\right)\left(\frac{m}{R} - bJ\right) + N_R - N_L - \gamma J \\ &= -\left[mw - wRbJ - \frac{\alpha' m\tilde{b}}{R}J + \alpha' b\tilde{b}J^2\right] + N_R - N_L - \gamma J \\ &= -mw + \gamma J + N_R - N_L - \gamma J \\ &= N_R - N_L - mw \end{aligned} \tag{2.98}$$

Therefore, the level matching condition is:

$$N_R - N_L = mw \tag{2.99}$$

The Hamiltonian (2.97) is a periodic function of the parameter γ, which also implies periodicity in the magnetic field strength parameters b and $\tilde{b}$.

The Hamiltonian was derived for γ in the range $0 \leq \gamma \leq 1$. For γ outside this range, because of a change in normal ordering constants, an integer must be added to γ so that it satisfies this condition. From the form (2.73) of γ, it can be seen that this implies invariance of the spectrum under the shift

$$b \rightarrow b + \frac{n}{R} \tag{2.100}$$

for any integer n, and that, in (2.97), b should be restricted to the interval

$$0 \leq b \leq \frac{1}{R} \tag{2.101}$$

This periodicity in b is to be expected, since, from the form of the KK Melvin metric (1.2), the background space is such that a translation in the x_9 direction is accompanied by a rotation in a plane by an angle $2\pi bR$, where R is the radius of the compact direction. The shift (2.100) thus adds $2\pi n$ to this angle, leaving the theory unchanged. Similarly, the spectrum is periodic under the shift

$$\tilde{b} \rightarrow \tilde{b} + \frac{nR}{\alpha'} \tag{2.102}$$

Consequently, $\tilde{b}$ in (2.97) should be restricted to the interval

$$0 \leq \tilde{b} \leq \frac{R}{\alpha'} \tag{2.103}$$

This periodicity also follows from the periodicity in b, since $\tilde{b}$ is the equivalent of b in the T-dualized space, for which the radius of the compact dimension is $\frac{\alpha'}{R}$.

2.2 Type II Superstring

The results of section 2.1 for the closed bosonic string can be extended to the type II superstring. This can be done by constructing a worldsheet supersymmetric version of the action (2.2) [3]. In conformal gauge, the RNS superstring Lagrangian has the form

$$\begin{aligned} L &= \frac{1}{1+\tilde{b}^2 x' x'^*}\left[\partial_+ u - \frac{\tilde{b}}{2i}\left(x'^*\partial_+ x' - x'\partial_+ x'^* - 2i\lambda'^*_L\lambda'_L\right)\right]\left[\partial_- v\right. \\ &\left. + \frac{\tilde{b}}{2i}\left(x'^*\partial_- x' - x'\partial_- x'^* - 2i\lambda'^*_R\lambda'_R\right)\right] + \partial_+ x'\partial_- x'^* + i\lambda'^*_R\partial_+\lambda'_R \\ &+ i\lambda'^*_L\partial_-\lambda'_L \end{aligned} \tag{2.104}$$

As in (2.16), the coordinate x is given by $x = re^{i\varphi} = x_1 + ix_2$, where x_1 and x_2 are cartesian coordinates in the $r - \varphi$ plane. λ_L and λ_R are the left- and right-moving components of the spinor, $\lambda = \lambda_1 + i\lambda_2$, corresponding to the x coordinate. The primes indicate the same rotation of the physical fields as in (2.9):

$$\begin{aligned} x' &= e^{i(bx_9+\tilde{b}t)}x \\ \lambda' &= e^{i(bx_9+\tilde{b}t)}\lambda \end{aligned} \tag{2.105}$$

The solution of this theory is similar to that of the bosonic theory. The equations of motion (2.19) and (2.20), for u and v become,

$$\begin{aligned} \partial_+ \frac{\partial L}{\partial(\partial_+ u)} &= 0 \\ \partial_+ \left[F(x')\left(\partial_- v + \tilde{b}A_- - \tilde{b}\lambda'^*_R\lambda'_R\right)\right] &= 0 \\ F(x')\left(\partial_- v + \tilde{b}A_- - \tilde{b}\lambda'^*_R\lambda'_R\right) &= h_- \end{aligned} \tag{2.106}$$

and

$$\begin{aligned} \partial_- \frac{\partial L}{\partial(\partial_- v)} &= 0 \\ \partial_- \left[F(x')\left(\partial_+ u - \tilde{b}A_+ + \tilde{b}\lambda'^*_L\lambda'_L\right)\right] &= 0 \\ F(x')\left(\partial_+ u - \tilde{b}A_- + \tilde{b}\lambda'^*_L\lambda'_L\right) &= h_+ \end{aligned} \tag{2.107}$$

The equation of motion, (2.21), for x' and its solution, (2.22), in the bosonic string case are unchanged in the superstring theory, except for the new form of h_+ and h_-, (2.106) and (2.107). Thus the solution of (2.106) and (2.107) for $\partial_- v$ and $\partial_+ u$ parallels (2.27) and (2.28), except for the addition of the fermionic term. Therefore,

$$\partial_+ u = \partial_+ U_+ + \frac{1}{2}i\tilde{b}\left(X\partial_+ X^* - X^*\partial_+ X\right) - \tilde{b}\lambda'^*_L\lambda'_L \tag{2.108}$$

$$\partial_- v = \partial_- V_- - \frac{1}{2}i\tilde{b}\left(X\partial_- X^* - X^*\partial_- X\right) + \tilde{b}\lambda'^*_R\lambda'_R \tag{2.109}$$

Integrating (2.108) and (2.109) gives

$$u = u_{bosonic} - \int_0^{\sigma_+} \tilde{b}\lambda'^*_L\lambda'_L d\sigma_+ + K_u(\sigma_-) \tag{2.110}$$

$$v = v_{bosonic} + \int_0^{\sigma_-} \tilde{b}\lambda'^*_R\lambda'_R d\sigma_- + K_v(\sigma_+) \tag{2.111}$$

where $u_{bosonic}$ and $v_{bosonic}$ are the corresponding solutions for the bosonic string, (2.29), and K_u and K_v are arbitrary functions. Taking $K_u = \int_0^{\sigma_-} \tilde{b}\lambda'^*_R\lambda'_R d\sigma_-$ and $K_v = -\int_0^{\sigma_+} \tilde{b}\lambda'^*_L\lambda'_L d\sigma_+$, the solution in the superstring case becomes the same as that in the bosonic string case, except with fermionic contributions to the angular momentum currents:

$$J_\pm(\sigma_\pm) = \frac{i}{4\pi\alpha'}\int_0^{\sigma_\pm} d\sigma_\pm \Big(X_\pm\partial_\pm X^*_\pm - X^*_\pm\partial_\pm X_\pm$$
$$+2i\lambda'^*_{\substack{L\\R}}\lambda'_{\substack{L\\R}}\Big) \tag{2.112}$$

For the fermions, the equations of motion are given by

$$\begin{aligned}\partial_+\frac{\partial L}{\partial(\partial_+\lambda'_R)} - \frac{\partial L}{\partial\lambda'_R} &= 0\\ \partial_+\lambda'_R + i\tilde{b}h_+\lambda'_R &= 0\end{aligned} \tag{2.113}$$

and

$$\begin{aligned}\partial_-\frac{\partial L}{\partial(\partial_-\lambda'_L)} - \frac{\partial L}{\partial\lambda'_L} &= 0\\ \partial_-\lambda'_L - i\tilde{b}h_-\lambda'_L &= 0\end{aligned} \tag{2.114}$$

As can be checked by substitution, the solution is (c.f. (2.22) and (2.23)):

$$\begin{aligned}\lambda'_R &= e^{-i\tilde{b}U_+ + i\tilde{b}V_-}\Lambda_R\,, \quad \partial_+\Lambda_R = 0\\ \lambda'_L &= e^{-i\tilde{b}U_+ + i\tilde{b}V_-}\Lambda_L\,, \quad \partial_-\Lambda_L = 0\\ \Lambda_R &= e^{-2i\gamma\sigma_-}\eta_-\\ \Lambda_L &= e^{2i\gamma\sigma_+}\eta_+\end{aligned} \tag{2.115}$$

where η_+ and η_- are free fermionic fields with expansions

$$\begin{aligned}\eta_-^{(NS)} &= \sqrt{2\alpha'}\sum_{r\in Z+\frac{1}{2}} c_r e^{-2ir\sigma_-} \qquad (2.116)\\ \eta_-^{(R)} &= \sqrt{2\alpha'}\sum_{n\in Z} d_n e^{-2in\sigma_-}\end{aligned}$$

and similarly for $\eta_+^{(NS)}$ and $\eta_+^{(R)}$. Since the boundary conditions (2.32) are unchanged, the form of $U_\pm$ and $V_\pm$ is still (2.33), so that γ is the same as in

(2.39). Moreover, it can be shown using (2.104) that the form of E and p_{x_9}, (2.60) and (2.71), is unchanged, so that γ is again given by (2.73).

As in the bosonic string theory, the energy-momentum tensor reduces to the free string theory form:

$$\begin{aligned} T_{++} &= \partial_+ U_+ \partial_+ V_+ + \partial_+ X \partial_+ X^* + i\Lambda_L^* \partial_+ \Lambda_L \\ T_{--} &= \partial_- U_- \partial_- V_- + \partial_- X \partial_- X^* + i\Lambda_R^* \partial_- \Lambda_R \end{aligned} \tag{2.117}$$

The bosonic parts of the Virasoro operators L_0 and $\tilde{L}_0$ are given by (2.52) and (2.54). Substituting (2.115) and (2.116) into (2.117) gives the fermionic parts, L_0^f and $\tilde{L}_0^f$, of L_0 and $\tilde{L}_0$:

$$\begin{aligned} L_0^f &= \frac{1}{4\pi\alpha'} \int_0^\pi i\Lambda_R^* \partial_- \Lambda_R d\sigma \\ &= \frac{i}{4\pi\alpha'} \int_0^\pi \left[\eta_-^* \eta_- (-2i\gamma) + \eta_-^* \partial_- \eta_- \right] \\ &= \begin{cases} \sum_{r\in Z+\frac{1}{2}} (r+\gamma)\, c_r^* c_r\ , & NS\ sector \\ \sum_{n\in Z} (n+\gamma)\, d_n^* d_n\ , & R\ sector \end{cases} \end{aligned} \tag{2.118}$$

$$\tilde{L}_0^f = \begin{cases} \sum_{r\in Z+\frac{1}{2}} (r-\gamma)\, \tilde{c}_r^* \tilde{c}_r\ , & NS\ sector \\ \sum_{n\in Z} (n-\gamma)\, \tilde{d}_n^* \tilde{d}_n\ , & R\ sector \end{cases} \tag{2.119}$$

The classical Hamiltonian is given by (2.56) with (2.118) and (2.119) added.

Quantization of the theory in the light-cone gauge leads to the same Hamiltonian, (2.97), except with fermionic contributions to N and J. From the Lagrangian (2.104), the anticommutation relations for the right-moving fermion fields are,

$$\left\{ \lambda_R^{*\prime}(\sigma,\tau), \lambda_R'(\sigma',\tau) \right\} = 2\pi\alpha' \delta(\sigma - \sigma') \tag{2.120}$$

From (2.115), this anticommutator is the same as that of the free fields, (2.116), since the phase factors cancel. Therefore, the oscillator commutation relations are the same as those of free string theory,

$$\begin{aligned} \{c_r^*, c_l\} &= \delta_{rl} \\ \{d_n^*, d_m\} &= \delta_{nm} \end{aligned} \tag{2.121}$$

The quantum version of L_0^f and $\tilde{L}_0^f$ is obtained from (2.118) and (2.119) by antisymmetrizing the oscillator products. For L_0^f in the Ramond sector, this gives:

$$\begin{aligned} L_0^f &= \frac{1}{2}\sum_{n=-\infty}^{\infty}(n+\gamma)\left(d_n^* d_n - d_n d_n^*\right) \\ &= \frac{1}{2}\sum_{n=1}^{\infty}(n+\gamma)\left(d_n^* d_n - d_n d_n^*\right) - \frac{1}{2}\sum_{n=1}^{\infty}(n-\gamma)\cdot \\ &\quad \left(d_{-n}^* d_{-n} - d_{-n} d_{-n}^*\right) + \frac{1}{2}\gamma\left[d_0^*, d_0\right] \end{aligned} \tag{2.122}$$

This expression can be normal-ordered using the expression for generalized zeta function regularization, (2.85). The result is

$$\begin{aligned} L_0^f &= \sum_{n=1}^{\infty}(n+\gamma)\, d_n^* d_n + \sum_{n=1}^{\infty}(n-\gamma)\, d_{-n} d_{-n}^* + \frac{1}{2}\gamma\left[d_0^*, d_0\right] \\ &\quad + \frac{1}{12} - \frac{1}{2}\gamma^2 \end{aligned} \tag{2.123}$$

A similar calculation can be used to derive the normal-ordered expressions for the left movers and for the NS sector. The normal-ordered level number operators are

$$N_R = \begin{cases} N_{bos} + \sum_{r=\frac{1}{2}}^{\infty} r\left(c_r^* c_r + c_{-r} c_{-r}^*\right), & NS\ sector \\ N_{bos} + \sum_{n=1}^{\infty} n\left(d_n^* d_n + d_{-n} d_{-n}^*\right), & R\ sector \end{cases} \tag{2.124}$$

and the same for N_L with tildes over the oscillators, where N_{bos} is the bosonic part, (2.89). The classical angular momentum operators are:

$$\begin{aligned} \hat{J}_R &= (\hat{J}_R)_{bos} + \frac{i}{4\pi\alpha'}\int_0^{\pi}(2i\lambda_R^*\lambda_R)\,d\sigma_- \\ &= (\hat{J}_R)_{bos} - \frac{1}{2\pi\alpha'}\int_0^{\pi}\eta_-^*\eta_-\,d\sigma_- \\ &= \begin{cases} \left(\hat{J}_R\right)_{bos} - \sum_{r\in Z+\frac{1}{2}} c_r^* c_r, & NS\ sector \\ \left(\hat{J}_R\right)_{bos} - \sum_{n\in Z} d_n^* d_n, & R\ sector \end{cases} \end{aligned} \tag{2.125}$$

and similarly for J_L. Antisymmetrizing and normal-ordering to obtain the quantum version,

$$\hat{J}_R = \begin{cases} \left(\hat{J}_R\right)_{bos} + \sum_{r\in Z+\frac{1}{2}}\left(c_r^* c_r - c_{-r} c_{-r}^*\right), & NS\ sector \\ \left(\hat{J}_R\right)_{bos} + \sum_{n=1}^{\infty}\left(d_n^* d_n - d_{-n} d_{-n}^*\right) + \frac{1}{2}\left[d_0^*, d_0\right], & R\ sector \end{cases} \tag{2.126}$$

When the fermionic parts of L_0 and $\tilde{L}_0$ are added to the bosonic contributions, the superstring Hamiltonian becomes

$$\begin{aligned} H &= -\frac{1}{2}\alpha' E^2 + N_L + N_R + \frac{1}{2}\alpha'\left(\frac{m}{R} - bJ\right)^2 + \\ &\frac{1}{2}\left(\frac{wR}{\sqrt{\alpha'}} - \sqrt{\alpha'}\tilde{b}J\right)^2 - \gamma\left(J_R - J_L\right) + c + \gamma \end{aligned} \tag{2.127}$$

where c is the normal-ordering constant of free type II superstring theory.

The range of γ in the superstring Hamitonian (2.127) is $-1 \leq \gamma \leq 1$, rather than $0 \leq \gamma \leq 1$ as in the bosonic theory. This is because superstring theory is only invariant under rotations by an even multiple of 2π, since fermions are multiplied by -1 when rotated by an odd multiple of 2π. Consequently, the shifts (2.100) and (2.102) under which b and $\tilde{b}$ are periodic become

$$\begin{aligned} b &\rightarrow b + \frac{2n}{R} \\ \tilde{b} &\rightarrow \tilde{b} + \frac{2nR}{\alpha'} \end{aligned} \tag{2.128}$$

From the form (2.73) of γ, this implies that γ is periodic only under even integer shifts.

Chapter 3
The Electric Melvin Solution

3.1 The Electric Melvin Background

The KK-Melvin metric

$$ds_{10}^2 = -dt^2 + dx_s^2 + dx_9^2 + dr^2 + r^2 (d\varphi + b dx_9)^2 \tag{3.1}$$

is equivalent to an orbifold of Minkowski space which identifies points under a combination of a translation in the x_9 direction and a rotation in the $r - \varphi$ plane [10]. This can be shown by introducing the coordinate $\varphi' = \varphi + bx_9$, giving

$$ds^2 = -dt^2 + dx_s^2 + dx_9^2 + dr^2 + r^2 d\varphi'^2 \tag{3.2}$$

or, using $x' = re^{i\varphi'}$,

$$ds^2 = -dt^2 + dx_s^2 + dx_9^2 + dx' dx'^* \tag{3.3}$$

This is Minkowski space with the identification

$$(x_9, \varphi') \equiv (x_9 + 2\pi R, \varphi' + 2\pi bR) \tag{3.4}$$

That is, points are identified under a rotation through the angle $2\pi bR$, accompanied by a translation by $2\pi R$ in the x_9 direction.

A natural generalization of this would be an orbifold of Minkowski space by a Lorentz boost, since, like rotations, Lorentz boosts are part of the Lorentz group of isometries of Minkowski space. A KK-Melvin-like boost orbifold can be constructed from a space involving a flat spatial direction X, a compact spatial direction x_9, and a time direction T. A boost in this space is given by

$$\begin{aligned} X' &= \frac{X - vT}{\sqrt{1 - v^2}} \\ T' &= \frac{T - vX}{\sqrt{1 - v^2}} \end{aligned} \tag{3.5}$$

where v is the velocity of the boosted frame. In light-cone coordinates $X^+ = X + T$, $X^- = X - T$, this transformation has the form

$$\begin{aligned} X^{+\prime} &= \sqrt{\frac{1-v}{1+v}} X^+ \\ X^{-\prime} &= \sqrt{\frac{1+v}{1-v}} X^- \end{aligned} \tag{3.6}$$

or, taking $e^\beta = \sqrt{\frac{1-v}{1+v}}$,

$$\begin{aligned} X^{+\prime} &= e^\beta X^+ \\ X^{-\prime} &= e^{-\beta} X^- \end{aligned} \tag{3.7}$$

The parameter β is an imaginary angle by which points in the $X - T$ plane are "rotated". This can be seen by dividing the $X - T$ plane into two regions as shown in Figure 3.1 and parametrizing the regions as follows:

$$\begin{aligned} I: \quad & X = tsinhx \\ & T = tcoshx \\ II: \quad & X = tcoshx \\ & T = tsinhx \end{aligned} \tag{3.8}$$

where $-\infty \leq t \leq \infty$ and $-\infty \leq x \leq \infty$. As shown in Figure 3.2, the value of t labels two hyperbolae, one in I and the other in II, while x labels points on these hyperbolae. In these coordinates, X^+ and X^- take the "polar" form

$$\begin{aligned} X^+ &= te^x \\ X^- &= \mp te^{-x} \end{aligned} \tag{3.9}$$

where the upper and lower signs refer to region I and region II, respectively. The Lorentz boost (3.7) corresponds to a shift of the imaginary angle x. The generalization of (3.3) and (3.4) to a boost orbifold is then

$$ds^2 = dX^{+\prime} dX^{-\prime} + dx_9^2 \tag{3.10}$$

$$(x_9, x') \equiv (x_9 + 2\pi R, x' + 2\pi E R) \tag{3.11}$$

where

$$\begin{aligned} X^{+\prime} &= te^{x'} \\ X^{-\prime} &= \mp te^{-x'} \\ x' &= x + E x_9 \end{aligned} \tag{3.12}$$

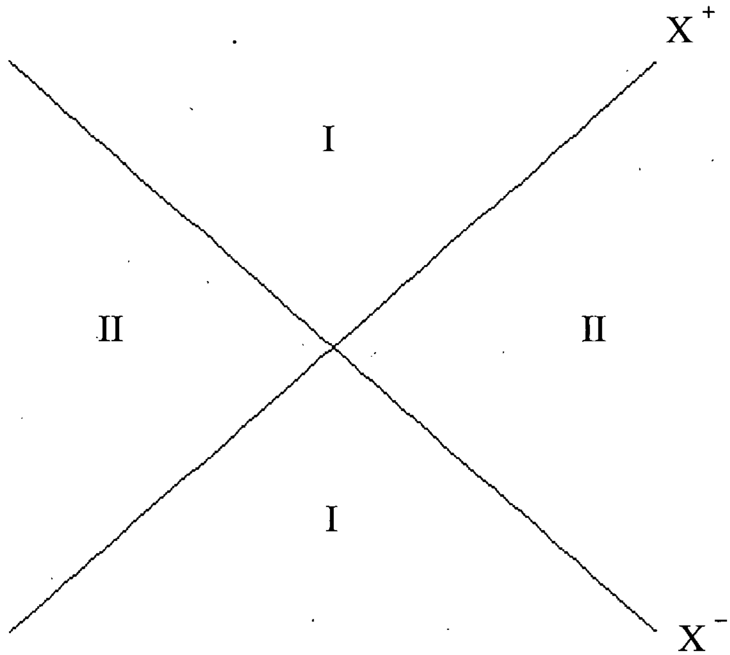

Figure 3.1: Regions I and II of the X-T plane.

and E is a constant. Points in this space are identified under a spatial translation in the x_9 direction accompanied by a spacetime translation along one of the hyperbolae in Figure 3.2. Substituting (3.12) into (3.10) gives

$$\begin{aligned} ds^2 &= d\left(te^{x+Ex_9}\right) d\left(\mp te^{-x-Ex_9}\right) + dx_9^2 \\ &= \left(dt + tdx + tEdx_9\right)\left(\mp dt \pm tdx \pm tEdx_9\right) + dx_9 \\ &= \mp dt^2 \pm t^2 \left(dx + Edx_9\right)^2 + dx_9^2 \end{aligned} \quad (3.13)$$

Flat Euclidean directions can be added to this space to attain the critical dimension for string theory, giving

$$ds^2 = dx_s^2 \mp dt^2 \pm t^2 \left(dx + Edx_9\right)^2 + dx_9^2 \quad (3.14)$$

As was done for the KK-Melvin background in chapter 1, dimensional reduction of the metric (3.14) may be carried out to produce a nine dimensional Kaluza-Klein theory. Rewriting (3.14) in the form (A.2) yields

$$ds^2 = dx_s^2 \mp dt^2 + \left(1 \pm E^2t^2\right)\left(dx_9 \pm \frac{Et^2}{1 \pm E^2t^2}dx\right)^2 \pm \quad (3.15)$$

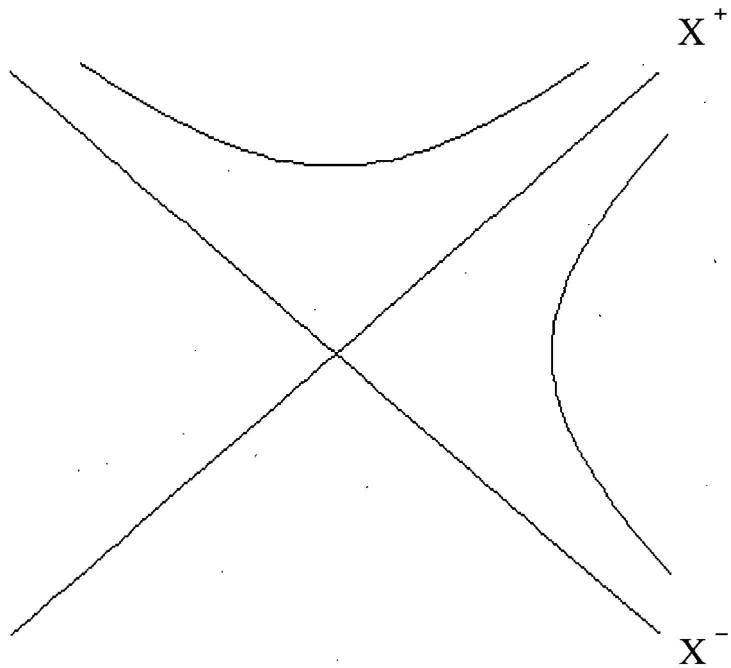

Figure 3.2: The coordinates t and x. Points with t equal to a positive constant and $-\infty < x < \infty$ are shown.

$$\frac{t^2}{1 \pm E^2 t^2} dx^2$$

Therefore, dimensional reduction gives

$$\begin{aligned} ds_9^2 &= dx_s^2 \mp dt^2 \pm \frac{t^2}{1 \pm E^2 t^2} dx^2 \\ A_x &= \pm \frac{E t^2}{1 \pm E^2 t^2} \qquad e^{2\sigma} = 1 \pm E^2 t^2 \end{aligned} \tag{3.16}$$

The nature of the gauge field can be made more clear by transforming to the cartesian coordinates X and T using (3.8). In these coordinates, the gauge field is

$$A_X = \frac{ET}{1 - E^2 (X^2 - T^2)} \qquad A_T = \frac{-EX}{1 - E^2 (X^2 - T^2)} \tag{3.17}$$

and the field strength is

$$F = dA$$

$$= \frac{2E}{\left[1 - E^2\left(X^2 - T^2\right)\right]^2} dT \wedge dX \tag{3.18}$$

This field strength is that of an electric field along the X-axis. The similarity of (3.14) to the Melvin background (3.1), and the fact that it gives rise to a background with an electric Kaluza-Klein gauge field on reduction to nine dimensions, suggests the interpretation of (3.14) as an electric KK-Melvin background.

Despite the similarities in the way in which the backgrounds (3.1) and (3.14) were constructed, the nature of the electric Melvin spacetime is unlike that of the magnetic Melvin background or of Melvin backgrounds in classical relativity. The electric Melvin background is not static in all regions, a property which is usually considered a defining characteristic of Melvin spaces [13]. It is static in region II, since, from (3.14), the metric has the timelike Killing vector $\frac{\partial}{\partial x}$, corresponding to a space-time translation along a hyperbola in the left or right wedge of Figure 3.2. However, in region I, there is no timelike Killing vector, and the space is time-dependent. Also, the electric Melvin background is pathological in some regions because of the existence of closed timelike curves (CTC's), which are unphysical because they violate causality. From (3.9) and the form of the electric Melvin metric (3.14), an infinitesimal translation $dx_9 = \frac{\beta}{E}$ is accompanied by an infinitesimal boost in the X direction given by

$$\begin{aligned} dX^+ &= \frac{\partial}{\partial x}\left(te^x\right)dx \\ &= \frac{\partial}{\partial x}\left(te^x\right)Edx_9 \\ &= \beta\frac{\partial}{\partial x}\left(te^x\right) \\ &= \beta te^x \\ dX^- &= \pm\beta te^{-x} \end{aligned} \tag{3.19}$$

This combined boost and translation becomes timelike in region II when the following condition is satisfied:

$$\begin{aligned} -dT^2 + dX^2 + dx_9^2 &< 0 \\ dX^+dX^- + dx_9^2 &< 0 \\ -\beta^2t^2 + \frac{\beta^2}{E^2} &< 0 \end{aligned}$$

$$|t| \; > \; \frac{1}{E} \tag{3.20}$$

For spacetime regions satisfying (3.20), any finite translation in x_9 with its accompanying boost becomes everywhere a timelike translation in spacetime, so that (3.11) identifies points separated by a timelike interval, and there exist CTC's connecting these points. Thus, these regions should be excluded when discussing string theory on the electric Melvin space. The regions are shown in Figure 3.3.

The electric KK-Melvin background can be generalized to a curved 2-parameter background in the same way as was done for the corresponding magnetic background in chapter 1. Starting with (3.14), a T-duality transformation along x_9 yields

$$\begin{aligned} ds^2 &= dx_s^2 \mp dt^2 + dx_9^2 \pm \frac{t^2}{1 \pm E^2t^2}\left(dx + Edx_9\right)\left(dx - Edx_9\right) \\ B_{xx_9} &= \pm\frac{Et^2}{1 \pm E^2t^2} \qquad e^{2(\Phi-\Phi_0)} = \frac{1}{1 \pm E^2t^2} \end{aligned} \tag{3.21}$$

The shift

$$x \to x + \tilde{E}dx_9 \tag{3.22}$$

introduces a second electric field parameter T-dual to E. This gives

$$\begin{aligned} ds^2 &= dx_s^2 \mp dt^2 + dx_9^2 \pm \frac{t^2}{1 \pm E^2t^2}\left[dx + \left(E + \tilde{E}\right)dx_9\right] \cdot \\ &\quad \left[dx + \left(\tilde{E} - E\right)dx_9\right] \\ B_{xx_9} &= \pm\frac{Et^2}{1 \pm E^2t^2} \qquad e^{2(\Phi-\Phi_0)} = \frac{1}{1 \pm E^2t^2} \end{aligned} \tag{3.23}$$

Another T-duality along x_9 gives

$$\begin{aligned} ds^2 &= dx_s^2 \mp dt^2 + dx_9^2 \pm \frac{t^2}{1 \pm \tilde{E}^2t^2}\left[dx + \left(E + \tilde{E}\right)dx_9\right] \cdot \\ &\quad \left[dx + \left(E - \tilde{E}\right)dx_9\right] \\ B_{xx_9} &= \pm\frac{\tilde{E}t^2}{1 \pm \tilde{E}^2t^2} \qquad e^{2(\Phi-\Phi_0)} = \frac{1}{1 \pm \tilde{E}^2t^2} \end{aligned} \tag{3.24}$$

When this background is dimensionally reduced, the resulting background has two electric gauge fields similar to that in (3.16) with parameters E and

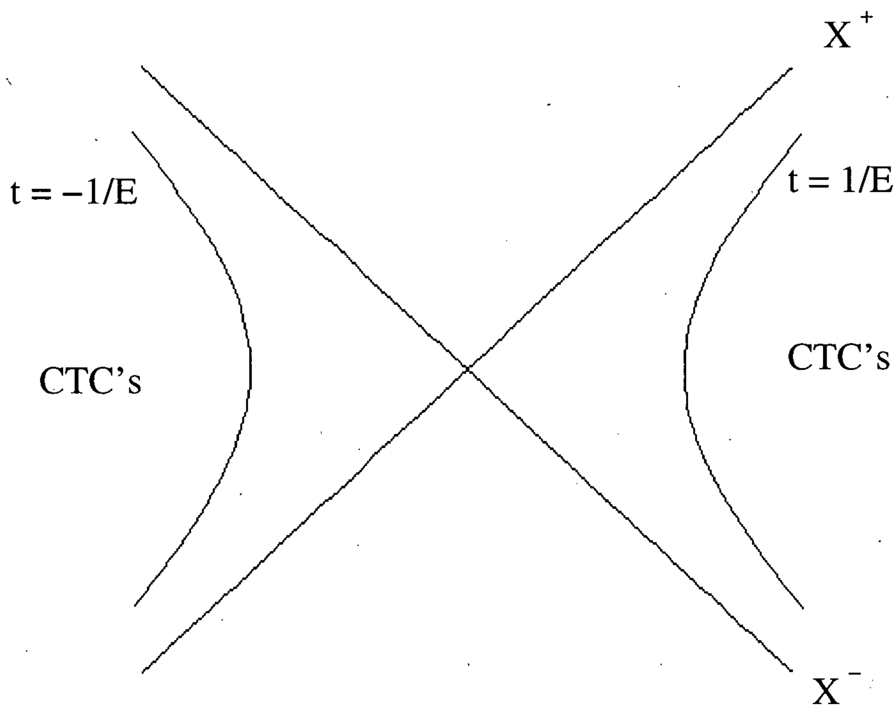

Figure 3.3: Regions with CTC's in the electric KK-Melvin space.

$\tilde{E}$. (3.24) can therefore be considered an electric Melvin background, like the background (3.14) of which it is a generalization. The properties of the spacetime (3.24) are similar to those of (3.14), but it has some additional pathologies. Timelike singularities occur in region II at $|t| = \frac{1}{\tilde{E}}$. This results from the fact that, at these points, the x_9 direction along which T-duality was carried out to produce (3.24) becomes null in (3.14). Also, the shift (3.22) results in an identification of points in the T-dualized space:

$$(x_9, x') \equiv \left(x_9 + 2\pi\frac{\alpha'}{R}, x' + 2\pi\tilde{E}\frac{\alpha'}{R}\right) \tag{3.25}$$

where $x' = x + \tilde{E}x_9$. This produces an additional region, $t \geq \frac{1}{\tilde{E}}$, where CTC's occur. These properties are illustrated in Figure 3.4.

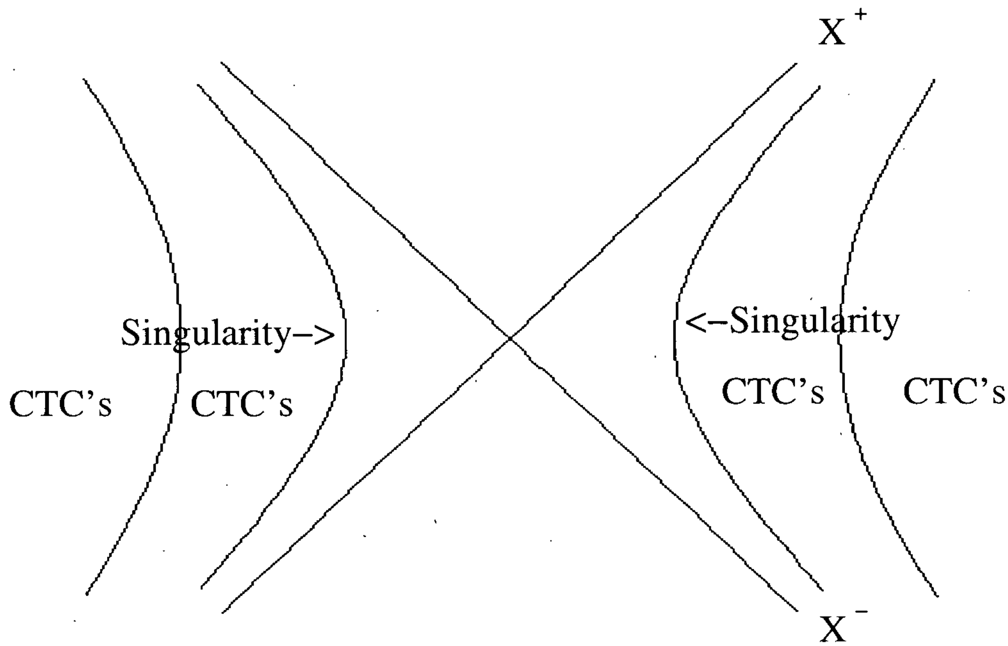

Figure 3.4: The 2-parameter electric Melvin background, for the case $\tilde{E} > E$. The inner curve is $|t| = \frac{1}{\tilde{E}}$, and the outer curve is $|t| = \frac{1}{E}$.

3.2 Quantization of the Closed String

Like its magnetic version, the electric Melvin background (3.24) admits an exact solution of closed string theory. Since the steps involved in the quantization are very similar to those in the magnetic case, only an outline will be given in this section.

In a suitably-chosen coordinate system, the Lagrangian reduces to a form similar to (2.16), for which the equations of motion may be solved in terms of free fields. The bosonic string Lagrangian for (3.24) is, omitting the flat space dimensions, x_s,

$$L = \mp\partial_+ t\partial_- t + \partial_+ x_9 \partial_- x_9 \pm \frac{t^2}{1 \pm \tilde{E}^2 t^2} \left[\partial_+ x + \left(E + \tilde{E}\right)\partial_+ x_9\right] \cdot \left[\partial_- x + \left(E - \tilde{E}\right)\partial_- x_9\right] \quad (3.26)$$

Introducing coordinates

$$x' = x + Ex_9$$

$$
\begin{aligned}
X^{+\prime} &= te^{x'} = e^{Ex_9}X^+ \\
X^{-\prime} &= \mp te^{-x'} = e^{-Ex_9}X^-
\end{aligned}
\tag{3.27}
$$

(3.26) reduces to

$$
\begin{aligned}
L &= \frac{1}{1-\tilde{E}^2X^+X^-}\left[\partial_+x_9 + \frac{\tilde{E}}{2}\left(X^{-\prime}\partial_+X^{+\prime} - X^{+\prime}\partial_+X^{-\prime}\right)\right]\cdot \\
&\left[\partial_-x_9 - \frac{\tilde{E}}{2}\left(X^{-\prime}\partial_-X^{+\prime} - X^{+\prime}\partial_-X^{-\prime}\right)\right] + \partial_+X^{+\prime}\partial_-X^{-\prime}
\end{aligned}
\tag{3.28}
$$

The light-cone coordinates u and v that were used in the magnetic case cannot be used in (3.28), because of the non-trivial time dependence. Defining

$$
\begin{aligned}
F\left(X^{-\prime}, X^{+\prime}\right) &= \frac{1}{1-\tilde{E}X^+X^-} \\
A_\pm &= \frac{1}{2}\left(X^{-\prime}\partial_\pm X^{+\prime} - X^{+\prime}\partial_\pm X^{-\prime}\right)
\end{aligned}
$$

the equations of motion of x_9, $X^{+\prime}$, and $X^{-\prime}$ become

$$F\left(\partial_+x_9 + \tilde{E}A_+\right) = h_+\left(\sigma_+\right) \tag{3.29}$$

$$F\left(\partial_-x_9 - \tilde{E}A_-\right) = h_-\left(\sigma_-\right) \tag{3.30}$$

$$\partial_+\partial_-X^{-\prime} + \tilde{E}h_-\partial_+X^{-\prime} - \tilde{E}h_+\partial_-X^{-\prime} - \tilde{E}^2h_+h_-X^{-\prime} = 0 \tag{3.31}$$

$$\partial_+\partial_-X^{+\prime} - \tilde{E}h_-\partial_+X^{+\prime} + \tilde{E}h_+\partial_-X^{+\prime} - \tilde{E}^2h_+h_-X^{+\prime} = 0 \tag{3.32}$$

where $h_+\left(\sigma_+\right)$ and $h_-\left(\sigma_-\right)$ are arbitrary functions. The solutions for $X^{-\prime}$ and $X^{+\prime}$ are

$$X^{-\prime} = e^{-\tilde{E}X_{9R}+\tilde{E}X_{9L}}Z^- \tag{3.33}$$

$$X^{+\prime} = e^{\tilde{E}X_{9R}-\tilde{E}X_{9L}}Z^+ \tag{3.34}$$

where X_{9L} and X_{9R} are functions of σ_+ and σ_- satisfying $\partial_-X_{9R} = h_-$ and $\partial_+X_{9L} = h_+$, and Z^- and Z^+ are free fields given by

$$
\begin{aligned}
Z^- &= Z_L^- + Z_R^- \qquad Z_L^- = e^{-2\gamma\sigma_+}\chi_L^- \qquad Z_R^- = e^{2\gamma\sigma_-}\chi_R^- \\
Z^+ &= Z_L^+ + Z_R^+ \qquad Z_L^+ = e^{2\gamma\sigma_+}\chi_L^+ \qquad Z_R^+ = e^{-2\gamma\sigma_-}\chi_R^+ \\
\chi_L^\pm &= \sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty}\tilde{a}_n^\pm exp\left(-2in\sigma_+\right) \\
\chi_R^\pm &= \sqrt{\frac{\alpha'}{2}}\sum_{n=-\infty}^{\infty}a_n^\pm exp\left(-2in\sigma_-\right)
\end{aligned}
\tag{3.35}
$$

The symbol Z was used instead of X as in chapter 2 to avoid confusion with the coordinates $X^{\pm}$. As in the magnetic case, the factors involving γ in (3.35) ensure periodicity of $X^{\pm}$. The solution for x_9 is

$$x_9 = X_{9L} + X_{9R} - \tilde{E}\tilde{\varphi} \tag{3.36}$$

where

$$\tilde{\varphi} = 2\pi\alpha' \left[J_R(\sigma_-) - J_L(\sigma_+)\right] + \frac{1}{2}\left(Z_L^+ Z_R^- - Z_R^+ Z_L^-\right) \tag{3.37}$$

and

$$J_{\substack{L\\R}} = \frac{1}{4\pi\alpha'}\int_0^{\sigma_\pm} d\sigma_\pm \left(Z^+_{\substack{L\\R}} \partial_\pm Z^-_{\substack{L\\R}} - Z^-_{\substack{L\\R}} \partial_\pm Z^+_{\substack{L\\R}}\right) \tag{3.38}$$

The boost momentum is defined by $J_L(\pi) + J_R(\pi)$.

Periodicity of the spacetime coordinates establishes the form of X_{9L}, X_{9R} and γ. The boundary conditions

$$\begin{aligned} x_9(\sigma+\pi,\tau) &= x_9(\sigma,\tau) + 2\pi w R \\ \tilde{\varphi}(\sigma+\pi,\tau) &= \tilde{\varphi}(\sigma,\tau) - 2\pi\alpha' J \end{aligned} \tag{3.39}$$

imply

$$X_{9\substack{L\\R}} = \sigma_\pm p_{\substack{L\\R}} + X'_{9\substack{L\\R}} \tag{3.40}$$

where

$$\begin{aligned} X'_{9L} &= \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{1}{n} \tilde{a}_n^9 exp\left(-2in\sigma_+\right) \\ X'_{9R} &= \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{1}{n} a_n^9 exp\left(-2in\sigma_-\right) \\ p_L &= \left(wR - \alpha'\tilde{E}J\right) + \frac{1}{2}p \\ p_R &= -\left(wR - \alpha'\tilde{E}J\right) + \frac{1}{2}p \end{aligned} \tag{3.41}$$

p is a constant which, by calculating the canonical momentum of the string in the x_9 direction, can be shown to be

$$p = 2\alpha'\left[\frac{m}{R} - EJ\right] \tag{3.42}$$

where m is the momentum number. Periodicity of X^+ and X^- implies

$$\begin{aligned} \gamma &= EwR + \frac{1}{2}\tilde{E}p \\ &= ERw + \frac{\tilde{E}\alpha' m}{R} - \alpha' E\tilde{E}J \end{aligned} \tag{3.43}$$

Unlike in the magnetic case, γ is not periodic, and there is no restriction on its values. This is because it appears in (3.35) in scaling factors, rather than in phase factors as in (2.23). It can also be seen from the form of the metric (3.14). When the periodic coordinate φ in (3.1) is replaced by the non-compact coordinate x in (3.14), the metric is no longer invariant under shifts of the field parameters such as (2.100).

The classical Hamiltonian is calculated as in chapter 2. The energy-momentum tensor reduces to the form

$$T_{\pm\pm} = \left(\partial_\pm X_{9\substack{L\\R}}\right)^2 + \partial_\pm Z^+ \partial_\pm Z^- \tag{3.44}$$

Substituting the solutions (3.35) into (3.44) gives

$$\begin{aligned} T_{--} &= p_R^2 + 2p_R\partial_- X'_{9R} + (\partial_- X'_{9R})^2 - 4\gamma^2\chi_R^+\chi_R^- - 2\gamma\left(\chi_R^+\partial_-\chi_R^- - \chi_R^-\partial_-\chi_R^+\right) \\ &+ \partial_-\chi_R^+\partial_-\chi_R^- \\ T_{++} &= p_L^2 + 2p_L\partial_+ X'_{9L} + (\partial_+ X'_{9L})^2 - 4\gamma^2\chi_L^+\chi_L^- - 2\gamma\left(\chi_L^-\partial_+\chi_L^+ - \chi_L^+\partial_+\chi_L^-\right) \\ &+ \partial_+\chi_L^+\partial_+\chi_L^- \end{aligned} \tag{3.45}$$

For the Virasoro operator L_0, this gives

$$\begin{aligned} L_0 &= \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma T_{--} \\ &= \frac{p_R^2}{4\alpha'} + \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma \left\{ \frac{\alpha'}{2}\left[\sum_{n=-\infty}^{\infty} a_n^9 (-2i)\, exp(-2in\sigma_-)\right]^2 - \right. \\ & 4\gamma^2\frac{\alpha'}{2}\left[\sum_{n=-\infty}^{\infty} a_n^+ exp(-2in\sigma_-)\right]\left[\sum_{n=-\infty}^{\infty} a_n^- exp(-2in\sigma_-)\right] - \\ & 2\gamma\left[\frac{\alpha'}{2}\left(\sum_{n=-\infty}^{\infty} a_n^+ exp(-2in\sigma_-)\right)\left(\sum_{n=-\infty}^{\infty} a_n^- (-2in)\, exp(-2in\sigma_-)\right) - \right. \\ & \left.\frac{\alpha'}{2}\left(\sum_{n=-\infty}^{\infty} a_n^- exp(-2in\sigma_-)\right)\left(\sum_{n=-\infty}^{\infty} a_n^+ (-2in)\, exp(-2in\sigma_-)\right)\right] + \end{aligned}$$

$$
\begin{aligned}
& \frac{\alpha'}{2}\left[\sum_{n=-\infty}^{\infty} a_n^+ (-2in)\, exp\,(-2in\sigma_-)\right]\left[\sum_{n=-\infty}^{\infty} a_n^- (-2in)\, exp\,(-2in\sigma_-)\right]\Bigg\} \\
= & \frac{p_R^2}{4\alpha'} + \frac{1}{2}\sum_{n=-\infty}^{\infty} a_n^9 a_{-n}^9 - \frac{1}{2}\gamma^2 \sum_{n=-\infty}^{\infty} a_n^+ a_{-n}^- - \\
& i\gamma \sum_{n=-\infty}^{\infty} n a_n^+ a_{-n}^- + \frac{1}{2}\sum_{n=-\infty}^{\infty} n^2 a_n^+ a_{-n}^- \\
= & \frac{p_R^2}{4\alpha'} + \frac{1}{2}\sum_{n=-\infty}^{\infty} a_n^9 a_{-n}^9 + \frac{1}{2}\sum_{n=-\infty}^{\infty} (n - i\gamma)^2\, a_n^+ a_{-n}^- \qquad (3.46)
\end{aligned}
$$

The second term, which is the oscillator contribution from x_9, was not present in the magnetic case because these oscillators were eliminated by the choice of light-cone gauge. The light-cone gauge cannot be used in the electric case, because of the non-trivial time dependence of the Lagrangian. The expression for $\tilde{L}_0$ is

$$
\tilde{L}_0 = \frac{p_L^2}{4\alpha'} + \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{a}_n^9 \tilde{a}_{-n}^9 + \frac{1}{2}\sum_{n=-\infty}^{\infty} (n + i\gamma)^2\, \tilde{a}_n^+ \tilde{a}_{-n}^- \qquad (3.47)
$$

Therefore, the Hamiltonian is:

$$
\begin{aligned}
H = & L_0 + \tilde{L}_0 \\
= & \frac{1}{4\alpha'}\left[-\left(wR - \alpha'\tilde{E}J\right) + \frac{1}{2}p\right]^2 + \frac{1}{4\alpha'}\left[\left(wR - \alpha'\tilde{E}J\right) + \frac{1}{2}p\right]^2 + \\
& \frac{1}{2}\sum_{n=-\infty}^{\infty} a_n^9 a_{-n}^9 + \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{a}_n^9 \tilde{a}_{-n}^9 + \\
& \frac{1}{2}\sum_{n=-\infty}^{\infty} (n - i\gamma)^2\, a_n^+ a_{-n}^- + \frac{1}{2}\sum_{n=-\infty}^{\infty} (n + i\gamma)^2\, \tilde{a}_n^+ \tilde{a}_{-n}^- \\
= & \frac{1}{8\alpha'}\left(4w^2R^2 + p^2\right) + \frac{1}{2}\sum_{n=-\infty}^{\infty} a_n^9 a_{-n}^9 + \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{a}_n^9 \tilde{a}_{-n}^9 + \qquad (3.48) \\
& \frac{1}{2}\sum_{n=-\infty}^{\infty} (n - i\gamma)^2\, a_n^+ a_{-n}^- + \frac{1}{2}\sum_{n=-\infty}^{\infty} (n + i\gamma)^2\, \tilde{a}_n^+ \tilde{a}_{-n}^- - \\
& \tilde{E}wRJ + \frac{1}{2}\alpha'\tilde{E}^2J^2
\end{aligned}
$$

The usual contribution from the 23 flat space coordinates should be added to this Hamiltonian.

Quantization of this theory is similar to that of the magnetic Melvin theory, except that the light-cone gauge cannot be used because of the non-linear time dependence. BRST quantization can be used instead. The canonical commutation relation is

$$\left[\frac{1}{4\pi\alpha'}\partial_\tau Z^+(\sigma,\tau), Z^-(\sigma',\tau)\right] = \left[\frac{1}{4\pi\alpha'}\partial_\tau Z^-(\sigma,\tau), Z^+(\sigma',\tau)\right] = -i\delta(\sigma-\sigma') \tag{3.49}$$

The left-hand side is

$$\begin{aligned}
&\frac{1}{4\pi\alpha'}\left[\partial_\tau Z^+(\sigma,\tau), Z^-(\sigma',\tau)\right] \\
=\ &\frac{1}{4\pi}\Big\{e^{2\gamma(\tau+\sigma)}\sum_{n=-\infty}^{\infty}(\gamma-in)\,\tilde{a}_n^+ e^{-2in(\tau+\sigma)} + e^{-2\gamma(\tau-\sigma)}. \\
&\sum_{n=-\infty}^{\infty}(-\gamma-in)\,a_n^+ e^{-2in(\tau-\sigma)}, e^{-2\gamma(\tau+\sigma)}\sum_{n=-\infty}^{\infty}\tilde{a}_n^- e^{-2in(\tau+\sigma)} + \\
&e^{2\gamma(\tau-\sigma)}\sum_{n=-\infty}^{\infty}a_n^- e^{-2in(\tau-\sigma)}\Big\} \\
=\ &\frac{-i}{4}e^{2\gamma(\sigma-\sigma')}\Big\{\frac{1}{\pi}\sum_{n=-\infty}^{\infty}e^{-2in(\sigma-\sigma')}\left[\tilde{a}_n^+, \tilde{a}_{-n}^-\right](n+i\gamma) + \\
&\frac{1}{\pi}\sum_{n=-\infty}^{\infty}e^{2in(\sigma-\sigma')}\left[a_n^+, a_{-n}^-\right](n-i\gamma)\Big\}
\end{aligned} \tag{3.50}$$

This leads to the commutators

$$\left[a_n^+, a_{-m}^-\right] = 2(n-i\gamma)^{-1}\delta_{n+m} \tag{3.51}$$

$$\left[\tilde{a}_n^+, \tilde{a}_{-m}^-\right] = 2(n+i\gamma)^{-1}\delta_{n+m} \tag{3.52}$$

The normalized creation and annihilation operators are:

$$\begin{aligned}
b_{n+}^\dagger &= \omega_+ a_{-n}^+ \quad & b_{n+} = \omega_+ a_n^- \\
b_{n-}^\dagger &= \omega_- a_{-n}^- \quad & b_{n-} = \omega_- a_n^+ \\
\tilde{b}_{n+}^\dagger &= \omega_- \tilde{a}_{-n}^+ \quad & \tilde{b}_{n+} = \omega_- \tilde{a}_n^- \\
\tilde{b}_{n-}^\dagger &= \omega_+ \tilde{a}_{-n}^- \quad & \tilde{b}_{n-} = \omega_+ \tilde{a}_n^+ \\
b_0^\dagger &= \sqrt{\frac{1}{2}i\gamma a_0^+} \quad & b_0 = \sqrt{\frac{1}{2}i\gamma a_0^-}
\end{aligned} \tag{3.53}$$

$$\begin{aligned}
\tilde{b}_0^\dagger &= \sqrt{\frac{1}{2} i\gamma \tilde{a}_0^-} \quad \tilde{b}_0 = \sqrt{\frac{1}{2} i\gamma \tilde{a}_0^+} \\
\omega_\pm &= \sqrt{\frac{1}{2}(n \pm i\gamma)}, \qquad n = 1, 2, \ldots
\end{aligned}$$

The b operators satisfy the commutation relations (2.81).

The commutation relations (3.51) and (3.52) can be used to normal-order the expression (3.48) to produce a quantum Hamiltonian. By symmetrizing and normal-ordering, it can be shown that

$$\frac{1}{2}\sum_n (n - i\gamma)^2 a_n^+ a_{-n}^- = \sum_{n=1}^{\infty} (n - i\gamma) b_{n-}^\dagger b_{n-} + \sum_{n=1}^{\infty} (n + i\gamma) b_{n+}^\dagger b_{n+} + i\gamma b_0^\dagger b_0 - \frac{1}{12} + \frac{1}{2} i\gamma (1 - i\gamma) \tag{3.54}$$

$$\frac{1}{2}\sum_n (n + i\gamma)^2 \tilde{a}_n^+ \tilde{a}_{-n}^- = \sum_{n=1}^{\infty} (n + i\gamma) \tilde{b}_{n-}^\dagger \tilde{b}_{n-} + \sum_{n=1}^{\infty} (n - i\gamma) \tilde{b}_{n+}^\dagger \tilde{b}_{n+} + i\gamma \tilde{b}_0^\dagger \tilde{b}_0 - \frac{1}{12} + \frac{1}{2} i\gamma (1 - i\gamma) \tag{3.55}$$

The quantum Hamiltonian is therefore

$$\begin{aligned}
H = {} & \frac{1}{8\alpha'}\left(4w^2R^2 + p^2\right) + \sum_{n=1}^{\infty} a_{-n}^9 a_n^9 + \sum_{n=1}^{\infty} \tilde{a}_{-n}^9 \tilde{a}_n^9 + \\
& \sum_{n=1}^{\infty} (n - i\gamma) b_{n-}^\dagger b_{n-} + \sum_{n=1}^{\infty} (n + i\gamma) b_{n+}^\dagger b_{n+} + i\gamma b_0^\dagger b_0 + \\
& \sum_{n=1}^{\infty} (n + i\gamma) \tilde{b}_{n-}^\dagger \tilde{b}_{n-} + \sum_{n=1}^{\infty} (n - i\gamma) \tilde{b}_{n+}^\dagger \tilde{b}_{n+} + i\gamma \tilde{b}_0^\dagger \tilde{b}_0 - \\
& \tilde{E} wRJ - 2 + i\gamma (1 - i\gamma) + \frac{1}{2}\alpha' \tilde{E}^2 J^2
\end{aligned} \tag{3.56}$$

This can also be expressed in terms of level number operators and boost operators. The level operators are given by

$$\begin{aligned}
N_R &= \sum_{n=1}^{\infty} \left[n\left(b_{n-}^\dagger b_{n-} + b_{n+}^\dagger b_{n+}\right) + a_{-n}^9 a_n^9\right] \\
N_L &= \sum_{n=1}^{\infty} \left[n\left(\tilde{b}_{n-}^\dagger \tilde{b}_{n-} + \tilde{b}_{n+}^\dagger \tilde{b}_{n+}\right) + \tilde{a}_{-n}^9 \tilde{a}_n^9\right]
\end{aligned} \tag{3.57}$$

From (3.38), the boost generators have the form

$$J_L = i\sum_{n=1}^{\infty}\tilde{b}^{\dagger}_{n-}\tilde{b}_{n-} - i\sum_{n=1}^{\infty}\tilde{b}^{\dagger}_{n+}\tilde{b}_{n+} + i\tilde{b}^{\dagger}_{0}\tilde{b}_{0} \tag{3.58}$$

$$J_R = i\sum_{n=1}^{\infty}b^{\dagger}_{n-}b_{n-} - i\sum_{n=1}^{\infty}b^{\dagger}_{n+}b_{n+} - ib^{\dagger}_{0}b_{0} \tag{3.59}$$

Substituting (3.57), (3.58), and (3.59) into (3.56),

$$H = N_L + N_R + \frac{1}{2}\alpha'\left(\frac{m}{R} - EJ\right)^2 + \frac{1}{2}\left(\frac{wR}{\sqrt{\alpha'}} - \sqrt{\alpha'}\tilde{E}J\right)^2 - \gamma\left(J_R - J_L\right) - 2 + i\gamma\left(1 - i\gamma\right) \tag{3.60}$$

The level-matching condition, $L_0 - \tilde{L}_0 = 0$, is

$$N_R - N_L = mw \tag{3.61}$$

These results can be extended to the type II superstring. The RNS superstring Lagrangian is

$$L = \frac{1}{1-\tilde{E}^2X^+X^-}\left[\partial_+ x_9 + \frac{\tilde{E}}{2}\left(X^{-\prime}\partial_+X^{+\prime} - X^{+\prime}\partial_+X^{-\prime} - 2i\lambda_L^{-\prime}\lambda_L^{+\prime}\right)\right]\left[\partial_- x_9 - \frac{\tilde{E}}{2}\left(X^{-\prime}\partial_-X^{+\prime} - X^{+\prime}\partial_-X^{-\prime} - 2i\lambda_R^{-\prime}\lambda_R^{+\prime}\right)\right] + \partial_+X^{+\prime}\partial_-X^{-\prime} + i\lambda_R^{-\prime}\partial_+\lambda_R^{+\prime} + i\lambda_L^{-\prime}\partial_-\lambda_L^{+\prime} \tag{3.62}$$

where $\lambda_L^{\pm}$ and $\lambda_R^{\pm}$ are the left- and right-moving components of the spinors $\lambda^{\pm} = \lambda_X \pm \lambda_T$, which correspond to the coordinates $X^{\pm}$, and

$$\lambda^{\pm\prime} = e^{\pm Ex_9}\lambda^{\pm} \tag{3.63}$$

Quantization of this theory leads to a Hamiltonian with the same form as (3.60), except with the operators N and J replaced with their fermionic versions, and the normal-ordering constant changed to $c + i\gamma$, where c is the normal-ordering constant of free superstring theory.

3.3 Partition Function

The partition function for the bosonic string in the electric Melvin background can be calculated from the usual expression in terms of a trace over states of the string:

$$Z = \int \frac{d^2\tau}{\tau_2} \int \prod_{a=1}^{23} dp_a \sum_{m,w=-\infty}^{\infty} Trexp\left[2\pi i\left(\tau L_0 - \bar{\tau}\tilde{L}_0\right)\right] \tag{3.64}$$

where $\tau = \tau_1 + i\tau_2$, τ_1 and τ_2 are the modular parameters of the torus, p_a are the momenta in the 23 free directions, and m and w are the momentum and winding modes in the x_9 direction. The domain of integration for τ_1 and τ_2 is the fundamental region,

$$F_0 = \left\{\tau_1, \tau_2 | \, |\tau| > 1, \; \frac{-1}{2} < \tau_1 < \frac{1}{2}, \; 0 < \tau_2 < \infty\right\} \tag{3.65}$$

(3.64) can be expressed as

$$Z = \int \frac{d^2\tau}{\tau_2} Z_{23} Z_g Z_3 \tag{3.66}$$

where Z_{23} and Z_g are the usual contributions of 23 free bosons and of the BRST ghosts, respectively, and Z_3 is the contribution of the three non-trivial coordinates, X^+, X^-, and x_9. Z_3 is given by

$$Z_3 = \sum_{m,w=-\infty}^{\infty} Trexp\left[2\pi i\tau_1\left(L_0 - \tilde{L}_0\right) - 2\pi\tau_2 H\right] \tag{3.67}$$

where H is given by (3.60) and $L_0 - \tilde{L}_0 = N_R - N_L - mw$.

The calculation of Z_3 is facilitated by expressing H in the form

$$\begin{aligned} H \;=\; & \frac{1}{2\alpha'}w^2R^2 + N_R + N_L + \frac{\alpha'}{2}\left\{\frac{m}{R} - \left[\left(E+\tilde{E}\right)J_R + \left(E-\tilde{E}\right)\right.\right. \\ & \left.\left. J_L\right]\right\}^2 - wR\left[\left(E+\tilde{E}\right)J_R - \left(E-\tilde{E}\right)J_L\right] + 2\alpha'\tilde{E}^2 J_R J_L + \\ & \gamma^2 + i\gamma - 2 \end{aligned} \tag{3.68}$$

Substituting (3.68) into (3.67) gives

$$\begin{aligned} Z_3 \;=\; & \sum_{m,w=-\infty}^{\infty} Trexp\left\{-2\pi i\tau_1\left[mw + \tilde{N} - N\right] - 2\pi\tau_2\left[\frac{1}{2\alpha'}w^2R^2 + \right.\right. \\ & N_R + N_L + \frac{\alpha'}{2}\left(\frac{m}{R} - \left(\left(E+\tilde{E}\right)J_R + \left(E-\tilde{E}\right)J_L\right)\right)^2 - \\ & \left.\left. wR\left(\left(E+\tilde{E}\right)J_R - \left(E-\tilde{E}\right)J_L\right) + 2\alpha'\tilde{E}^2 J_R J_L + \gamma^2 + i\gamma - 2\right]\right\} \end{aligned} \tag{3.69}$$

This expression contains terms quadratic in J_L and J_R, which must be linearized before the trace can be calculated. The γ^2 term can be linearized using the identity

$$e^{-2\pi\tau_2\gamma^2} = \sqrt{2\tau_2}\int_{-\infty}^{\infty} d\nu e^{-2\pi\tau_2\nu^2-4\pi i\tau_2\gamma\nu} \tag{3.70}$$

(3.70) can be verified by completing the square in the exponent and evaluating the resulting Gaussian integral. From the form (3.60) of H, it can be seen that the remaining terms linear in γ in (3.69) and (3.70) can be absorbed into a redefinition of J_R and J_L:

$$\begin{aligned} J_R' &= J_R - i\nu - \frac{1}{2}i \\ J_L' &= J_L + i\nu + \frac{1}{2}i \end{aligned} \tag{3.71}$$

The contribution to (3.69) of the fourth term in (3.68) becomes linear in J_L and J_R after Poisson resummation over m is carried out. After completing the square, the m-dependent part of (3.69) becomes

$$\sum_{m,w=-\infty}^{\infty} exp - \left\{\pi\tau_2\alpha'\left[\frac{m}{R} + i\frac{wR\tau_1}{\alpha'\tau_2} - \left(E+\tilde{E}\right)J_R - \left(E-\tilde{E}\right)J_L\right]^2 + 2\pi iRw\tau_1\tau_2\left[\left(E+\tilde{E}\right)J_R + \left(E-\tilde{E}\right)J_L\right] + \frac{\pi R^2w^2\tau_1^2}{\alpha'\tau_2}\right\} \tag{3.72}$$

The Poisson resummation formula is [5]:

$$\sum_{m=-\infty}^{\infty} exp\left[\frac{-\pi\left(m-b\right)^2}{a}\right] = \sqrt{a}\sum_{w'=-\infty}^{\infty} exp\left(-\pi a w'^2 + 2\pi i b w'\right) \tag{3.73}$$

where a and b are arbitrary constants. Using (3.73), (3.72) becomes

$$\frac{R}{\sqrt{\alpha'\tau_2}}\sum_{w'=-\infty}^{\infty} exp\left\{-\frac{\pi R^2}{\alpha'\tau_2}w'^2 + \frac{2\pi R^2\tau_1}{\alpha'\tau_2}ww' + 2\pi iR\left[\left(E+\tilde{E}\right)J_R + \left(E-\tilde{E}\right)J_L\right]w' - 2\pi iR\tau_1\left[\left(E+\tilde{E}\right)J_R + \left(E-\tilde{E}\right)J_L\right]w - \frac{\pi R^2\tau_1^2}{\alpha'\tau_2}w^2\right\} \tag{3.74}$$

Substituting (3.70) and (3.74) into (3.69), and using (3.71), (3.69) reduces to

$$\begin{aligned} Z_3 \;=\; & \frac{\sqrt{2}R}{\sqrt{\alpha'}} \int_{-\infty}^{\infty} d\nu Trexp\left[-2\pi\tau_2\left(\nu^2-2\right)\right] \sum_{w,w'=-\infty}^{\infty} exp\left[-\frac{\pi R^2}{\alpha'\tau_2}\cdot\right. \\ & \left.(w'-\tau w)(w'-\bar{\tau}w)\right] exp\left[2\pi i\left(\tau N_R-\bar{\tau}N_L\right)\right] exp\left[2\pi i\left(w'-\tau w\right)\cdot\right. \\ & \left. R\left(E+\tilde{E}\right)J'_R\right] exp\left[2\pi i\left(w'-\bar{\tau}w\right)R\left(E-\tilde{E}\right)J'_L\right]\cdot \\ & exp\left(-4\pi\tau_2\alpha'\tilde{E}^2 J'_R J'_L\right) \end{aligned} \tag{3.75}$$

The last factor in (3.75) can be made linear in the boost operators by using the identity

$$\begin{aligned} 1 \;=\; & \frac{4}{\tau_2}\int d\lambda_1\lambda_2 exp\left\{-\frac{4\pi}{\tau_2}\left[\lambda-\frac{R}{2\sqrt{\alpha'}}\left(w'-\tau w\right)-\right.\right. \\ & \left.\left. i\tau_2\sqrt{\alpha'}\tilde{E}J'_L\right]\left[\bar{\lambda}+\frac{R}{2\sqrt{\alpha'}}\left(w'-\bar{\tau}w\right)-i\tau_2\sqrt{\alpha'}\tilde{E}J'_R\right]\right\} \end{aligned} \tag{3.76}$$

where $\lambda=\lambda_1+i\lambda_2$. This identity can be proved by converting the right-hand side into a product of Gaussian integrals over λ_1 and λ_2. When (3.76) is inserted into (3.75), the first and last factors in (3.75) cancel out, giving

$$\begin{aligned} Z_3 \;=\; & \frac{4\sqrt{2}R}{\sqrt{\alpha'}\tau_2}e^{4\pi\tau_2}\int_{-\infty}^{\infty} d\nu exp\left(-2\pi\tau_2\nu^2\right)\cdot \\ & \sum_{w,w'=-\infty}^{\infty}\int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 exp\left\{-\frac{4\pi}{\tau_2}\left[\lambda\bar{\lambda}-\frac{R}{2\sqrt{\alpha'}}\left(w'-\tau w\right)\bar{\lambda}\right.\right. \\ & \left.\left.+\frac{R}{2\sqrt{\alpha'}}\left(w'-\bar{\tau}w\right)\lambda\right]\right\} Trexp\left[2\pi i\left(\tau N_R-\chi J'_R\right)\right]\cdot \\ & Trexp\left[-2\pi i\left(\bar{\tau}N_L+\bar{\chi}J'_L\right)\right] \end{aligned} \tag{3.77}$$

where

$$\chi=-\sqrt{\alpha'}\left[2\tilde{E}\lambda+E\frac{R}{\sqrt{\alpha'}}\left(w'-\tau w\right)\right] \tag{3.78}$$

The traces in (3.77) can be evaluated using (3.57), (3.58) and (3.59). For the right-movers, this gives

$$Trexp\left[2\pi i\left(\tau N_R-\chi J'_R\right)\right]$$

$$
\begin{aligned}
&= Trexp2\pi i\left\{\tau\sum_{n=1}^{\infty}\left[n\left(b_{n-}^{\dagger}b_{n-}+b_{n+}^{\dagger}b_{n+}\right)+a_{-n}^{9}a_{n}^{9}\right]-\right.\\
&\quad \chi i\sum_{n=1}^{\infty}b_{n-}^{\dagger}b_{n-}+\chi i\sum_{n=1}^{\infty}b_{n+}^{\dagger}b_{n+}+\chi i b_0^{\dagger}b_0+\\
&\quad \left. i\nu\chi+\frac{1}{2}i\chi\right\}\\
&= e^{-2\pi\chi\left(\nu+\frac{1}{2}\right)}Trexp\left\{2\pi i\sum_{n=1}^{\infty}\left(n\tau-i\chi\right)b_{n-}^{\dagger}b_{n-}+\right.\\
&\quad \left.2\pi i\sum_{n=1}^{\infty}\left(n\tau+i\chi\right)b_{n+}^{\dagger}b_{n+}-2\pi\chi b_0^{\dagger}b_0+2\pi i\tau\sum_{n=1}^{\infty}a_{-n}^{9}a_{n}^{9}\right\}\\
&= e^{-2\pi\chi\left(\nu+\frac{1}{2}\right)}\left(\frac{1}{1-e^{-2\pi\chi}}\right)\prod_{n=1}^{\infty}\left(\frac{1}{1-e^{2\pi i(n\tau-i\chi)}}\right)\cdot\\
&\quad \left(\frac{1}{1-e^{2\pi i(n\tau+i\chi)}}\right)\left(\frac{1}{1-e^{2\pi i n\tau}}\right)
\end{aligned}
\tag{3.79}
$$

Similarly,

$$
\begin{aligned}
& Trexp\left[-2\pi i\left(\bar{\tau}N_L+\bar{\chi}J_L'\right)\right]\\
&= e^{2\pi\bar{\chi}\left(\nu+\frac{1}{2}\right)}\left(\frac{1}{1-e^{2\pi\bar{\chi}}}\right)\prod_{n=1}^{\infty}\left(\frac{1}{1-e^{-2\pi i(n\bar{\tau}+i\bar{\chi})}}\right)\cdot\\
&\quad \left(\frac{1}{1-e^{-2\pi i(n\bar{\tau}-i\bar{\chi})}}\right)\left(\frac{1}{1-e^{-2\pi i n\bar{\tau}}}\right)
\end{aligned}
\tag{3.80}
$$

The integral over ν in (3.77) can be evaluated using the ν-dependence of the traces given by (3.79) and (3.80). The result is:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}d\nu exp\left\{-2\pi\tau_2\nu^2-2\pi\chi\left(\nu+\frac{1}{2}\right)+2\pi\bar{\chi}\left(\nu+\frac{1}{2}\right)\right\}\\
&= \frac{1}{\sqrt{2\tau_2}}e^{\frac{\pi(\chi-\bar{\chi})^2}{2\tau_2}-\pi(\chi-\bar{\chi})}
\end{aligned}
\tag{3.81}
$$

Substituting (3.79), (3.80), and (3.81) into (3.77) gives

$$
\begin{aligned}
Z_3 &= -\frac{4R}{\sqrt{\alpha'}\tau_2^{\frac{3}{2}}}e^{\frac{7}{2}\pi\tau_2}\sum_{w,w'}^{\infty}\int_{-\infty}^{\infty}d\lambda_1 d\lambda_2 exp\left\{-\frac{4\pi}{\tau_2}\left[\lambda\bar{\lambda}-\right.\right.\\
&\quad \frac{R}{2\sqrt{\alpha'}}\left(w'-\tau w\right)\bar{\lambda}+\frac{R}{2\sqrt{\alpha'}}\left(w'-\bar{\tau}w\right)\lambda-\\
&\quad \left.\left.\frac{1}{8}\left(\chi-\bar{\chi}\right)^2\right]\right\}\frac{1}{|\Theta_1\left(i\chi,\tau\right)|^2}
\end{aligned}
\tag{3.82}
$$

where Θ_1 is the Jacobi Θ_1-function given by

$$\Theta_1(\nu,\tau) = 2exp\left(\frac{\pi i\tau}{4}\right) sin\pi\nu \prod_{n=1}^{\infty}\left(1-e^{2\pi i n\tau}\right)\cdot \left(1-e^{2\pi i(n\tau+\nu)}\right)\left(1-e^{2\pi i(n\tau-\nu)}\right) \quad (3.83)$$

Unlike the partition functions of the magnetic Melvin solution and of free string theory, which are analytic everywhere in the interior of the fundamental domain F_0, the electric Melvin partition function has an infinite number of simple poles in F_0. This can easily be seen in the special cases $\tilde{E} = 0$ and $E = 0$. Choosing $\tilde{E} = 0$, and considering only the terms with $w = 0$, the Θ_1-function in (3.82) becomes

$$\Theta_1(i\chi,\tau) = 2exp\left(\frac{\pi i\tau}{4}\right) sin\left(-i\pi ERw'\right) \prod_{n=1}^{\infty}\left(1-e^{2\pi i n\tau}\right)\cdot \left(1-e^{2\pi i(n\tau-iERw')}\right)\left(1-e^{2\pi i(n\tau+iERw')}\right) \quad (3.84)$$

The second factor in the infinite product can become zero at values of τ_1 and τ_2 satisfying

$$2\pi i\left(n\tau - iERw'\right) = 2\pi ik \quad (3.85)$$

for some integer k. Solving for τ gives

$$\tau = \frac{k}{n} + i\frac{ERw'}{n} \quad (3.86)$$

For values of k, n, and w' such that $|\frac{k}{n}| < \frac{1}{2}$, $w' > 0$, and

$$\frac{k^2}{n^2} + \frac{E^2R^2w'^2}{n^2} > 1$$

this zero lies in F_0. Thus, the factor of $\left[\Theta_1(i\chi,\tau)\right]^{-1}$ in (3.82) has an infinite number of poles in the fundamental domain. The presence of poles can also be demonstrated in the case $E = 0$, $\tilde{E} \neq 0$. In this case, the Θ_1-function is independent of w and w', and the sums over w and w' yield δ-functions, allowing the integrals over λ_1 and λ_2 to be evaluated. The resulting expression contains a Θ_1-function with zeros at the points

$$\tau = \frac{k}{n} + i\frac{\alpha'\tilde{E}w'}{nR} \quad (3.87)$$

for arbitrary integers k and w', producing poles in the partition function at these points.

Chapter 4
Conclusion

In this thesis, an electric Melvin solution of string theory analogous to the magnetic Melvin solution of [2] was presented. The electric KK-Melvin background described in [11] was generalized to a curved, 2-parameter background using a T-duality transformation. The closed string was quantized on this space using techniques similar to those used in [2] and [3] for the magnetic Melvin space. The partition function was calculated, and was shown to possess poles in the interior of the fundamental region for two special cases of the electric field parameters.

There are many possibilities for further research on the electric Melvin solution. More insight into the nature of the solution and its implications for string theory would be provided by determining the physical significance of the poles in the partition function described in section 3.3. Also, the electric field in this background should result in pair creation of strings by the Schwinger mechanism. It may be possible to find an expression for the pair production rate by calculating the imaginary part of the partition function, as was done for open strings in a constant electric field in [14]. Another possibility is to find generalizations of the solution. Other string theories, such as open string theories, could be quantized on the space. Combinations of the magnetic and electric Melvin solutions could also be constructed, and may be worth investigating. This could be done by quantizing the string on an orbifold which identifies points under both rotations and boosts.

Appendix A
Dimensional Reduction

Dimensional reduction is a technique by which the low-energy limit of string theory on a space with a compact dimension can be viewed as a gauge theory on a lower dimensional space [5]. For a $d+1$-dimensional space with dimensions x^M, $M = 0, 1, \ldots d$, with x^d compact, the line element can be expressed as

$$\begin{aligned} ds^2 &= g_{MN} dx^M dx^N \\ &= g_{\mu\nu} dx^\mu dx^\nu + 2 g_{\mu d} dx^\mu dx^d + g_{dd} \left(dx^d\right)^2 \end{aligned} \tag{A.1}$$

where M and N range from 0 to d, and μ and ν range from 0 to $d-1$. Completing the square, (A.1) can be rewritten in the form

$$ds^2 = g'_{\mu\nu} dx^\mu dx^\nu + g_{dd} \left(dx^d + A_\mu dx^\mu\right)^2 \tag{A.2}$$

for some vector A_μ. In this form, the original metric has been separated into a d-dimensional metric $g'_{\mu\nu}$, a scalar g_{dd}, called the Brans-Dicke scalar, and a vector field A_μ, called a Kaluza-Klein field.

If the metric g_{MN} of the $d+1$-dimensional space does not depend on x^d, this space can be viewed as a d-dimensional space with metric $g'_{\mu\nu}$ and gauge field A_μ. The metric $g'_{\mu\nu}$ transforms as a d-dimensional metric for coordinate transformations of the form $x^\mu \to x'^\mu \left(x^\nu\right)$. Under the transformation

$$x'^d = x^d + \lambda \left(x^\mu\right) \tag{A.3}$$

the metric (A.2) is invariant if the gauge transformation

$$A'_\mu = A_\mu - \partial_\mu \lambda \left(x^\mu\right) \tag{A.4}$$

is made. Fields which are charged under the Kaluza-Klein gauge field can occur in the low-energy string spectrum. For example, because of the identification $x^d \equiv x^d + 2\pi R$, a scalar field ϕ can be expanded as

$$\phi\left(x^M\right) = \sum_{n=-\infty}^{\infty} \phi_n \left(x^\mu\right) exp \left(\frac{inx^d}{R}\right) \tag{A.5}$$

Under the translation (A.3), a given mode in this expansion is multiplied by a factor of $e^{-\frac{in\lambda}{R}}$. The modes are thus charged under A_μ, with charges equal to the quantized momentum $\frac{n}{R}$ in the x^d direction.

This technique can be extended to backgrounds with an antisymmetric tensor field, B^{MN}. The general string action (2.1) is invariant under the gauge transformation

$$\delta B_{MN} = \partial_M \zeta_N - \partial_N \zeta_M \tag{A.6}$$

where ζ_μ is an arbitrary vector. This transformation adds a total derivative to the action. For ζ_M independent of x^d, (A.6) reduces to a gauge transformation of an antisymmetric tensor field and of a vector:

$$\begin{aligned} \delta B_{\mu\nu} &= \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu \\ \delta B_{\mu x^d} &= \partial_\mu \zeta_{x^d} \end{aligned} \tag{A.7}$$

Therefore, the dimensionally-reduced theory involves an antisymmetric tensor field, $B_{\mu\nu}$, and a one-form gauge field, $B_\mu = B_{\mu x^d}$.

Appendix B
T-duality

T-duality is a symmetry of 2-dimensional conformal field theory which relates different spacetime backgrounds in string theory. For a given background with metric $g_{\mu\nu}$, antisymmetric tensor $B_{\mu\nu}$, and dilaton Φ, which are independent of some coordinate x^0, the general bosonic string worldsheet action (2.1) is equivalent on shell to the action [16]

$$\begin{aligned} S = & -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \left\{\sqrt{h}h^{\alpha\beta}\left[g_{00}V_\alpha V_\beta + 2g_{0i}V_\alpha\partial_\beta X^i + \right.\right. \\ & \left. g_{ij}\partial_\alpha X^i\partial_\beta X^j\right] + \epsilon^{\alpha\beta}\left[B_{0i}V_\alpha\partial_\beta X^i + B_{ij}\partial_\alpha X^i\partial_\beta X^j\right] + \\ & \left.\epsilon^{\alpha\beta}\tilde{X}^0\partial_\alpha V_\beta + \alpha'\sqrt{h}R\Phi\left(X\right)\right\} \end{aligned} \tag{B.1}$$

where i and j run from 1 to 25. The field equation for $\tilde{X}^0$, $\epsilon^{\alpha\beta}\partial_\alpha V_\beta = 0$, has as a solution $V_\alpha = \partial_\alpha X^0$, which when substituted into (B.1) gives back (2.1). Another action which is equivalent to (B.1) on shell can be derived by solving for V_α through its equation of motion, and substituting the solution into (B.1). This gives an action of the form

$$\begin{aligned} S = & -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \left[\sqrt{h}\partial_\alpha\tilde{X}^\mu\partial_\beta\tilde{X}^\nu h^{\alpha\beta}\tilde{g}_{\mu\nu} + \epsilon^{\alpha\beta}\partial_\alpha\tilde{X}^\mu\partial_\beta\tilde{X}^\nu\tilde{B}_{\mu\nu}\right. \\ & \left. + \alpha'\sqrt{h}R\Phi\left(\tilde{X}\right)\right] \end{aligned} \tag{B.2}$$

with

$$\begin{aligned} \tilde{g}_{00} &= \frac{1}{g_{00}}, \qquad \tilde{g}_{0i} = \frac{B_{0i}}{g_{00}} \\ \tilde{g}_{ij} &= g_{ij} - \frac{g_{0i}g_{0j} - B_{0i}B_{0j}}{g_{00}} \\ \tilde{B}_{0i} &= \frac{g_{0i}}{g_{00}} \\ \tilde{B}_{ij} &= B_{ij} + \frac{g_{0i}B_{0j} - B_{0i}g_{0j}}{g_{00}} \end{aligned} \tag{B.3}$$

In addition to preserving the classical properties of the background, the transformation (B.3) also preserves conformal invariance at one-loop order, provided that it is accompanied by a shift in the dilaton:

$$\Phi \to \Phi - \frac{1}{2} ln g_{00} \tag{B.4}$$

This can be shown by substituting (B.3) and (B.4) into the one-loop conformal invariance equations of string theory [15]. This transformation is called a T-duality transformation along the X^0 direction.

When T-duality is done along a compact direction, a generalization is obtained of the $R \to \frac{1}{R}$ duality of string theory on flat space with a compact dimension of radius R [17]. For a space with $X^0 \equiv X^0 + 2\pi$, if the identification $\sqrt{g_{00}} \equiv R$ is made, the direction $\tilde{X}^0$ in the T-dual space is compact with radius $\frac{1}{\sqrt{g_{00}}}$. In the case of flat space, $g_{0i} = B_{\mu\nu} = 0$ and (B.3) reduces to the $R \to \frac{1}{R}$ duality under which the string spectrum is invariant.

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